Exponentials

- These expressions with x's in the power are actually functions called exponentials.

- We alkeady know some functions:

$y = 2x+1$ ← Line

$y = x^2$ ← Quadratic (Parabola)

Now $y = 2^x$ ← exponential.

$y = 2^x$ ← exp with base 2

$y = 3^x$ ← exp with base 3

$y = 10^x$

$y = \left(\frac{1}{2}\right)^x$

← These are all exponentials.

Exponential : $f(x) = b^x$ b = positive constant
$b \neq 1$.

(* b cant be negative)

Lets sketch $f(x) = 2^x$.

- Notice $2^0 = 1$ ie: when $x=0$, $y=1$

  Note this is true for any exp.

  $\therefore$ all exps cut y-axis at 1.

- as x gets bigger: $x=1$ $y=2$

  $x=2$ $y=4$

  $x=3$ $y=8$

  $\downarrow$ increasing fast.

- as x gets smaller: $x=-1$ $y=2^{-1}=\frac{1}{2}$

  $x=-2$ $y=2^{-2}=\frac{1}{4}$

  $\downarrow$ decreasing but never neg.

  $\vdots$

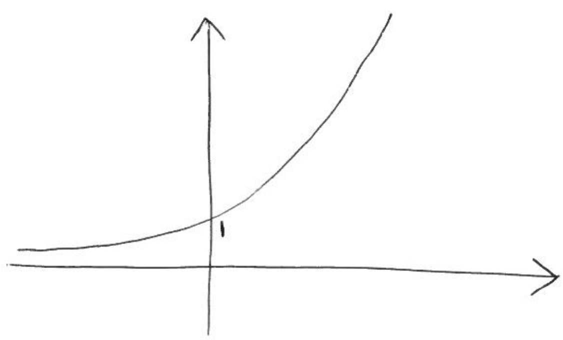
  $x=-100$ $y=\frac{1}{2^{100}}$

  $x=-\infty$ $y=\frac{1}{\infty}$ $\leftarrow$ very small.

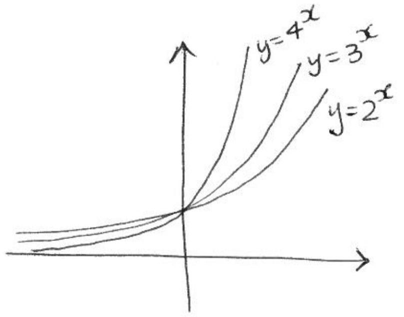
  correctly we should say

  as $x \to -\infty$, $y \to 0$

ie:

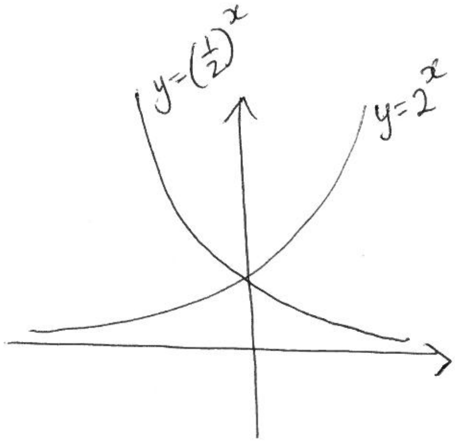


Different bases affect the steepness of the graph



- what about $y=\left(\frac{1}{2}\right)^x$.

x	2^x	$\left(\frac{1}{2}\right)^x$
2	4	1/4
1	2	1/2
0	0	0
-1	1/2	2
-2	1/4	4

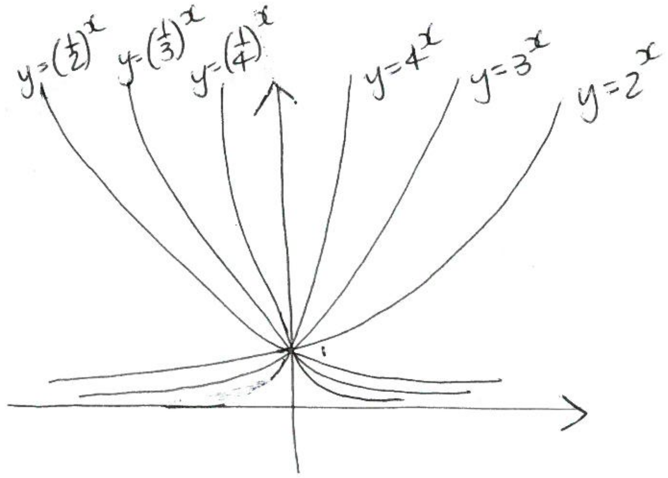


decreases rapidly
but is still never neg
doesn't cross x-axis.

Infact its a reflection of $f(x)=2^x$ about y axis.

Notice we can see that by looking at the equation

$$\begin{aligned} y &= \left(\tfrac{1}{2}\right)^x \\ &= (2^{-1})^x \\ &= 2^{-x} \quad \leftarrow f(-x) \end{aligned}$$



For exponentials $f(x) = b^x$ notice

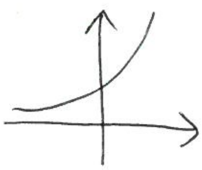
- All pass through $(0,1)$ ie: cuts y axis at $y=1$
- x axis is a horizontal asymptote

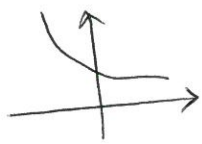
line function approaches but doesnt touch

- Dom $= \mathbb{R}$ + Range $= (0,\infty)$

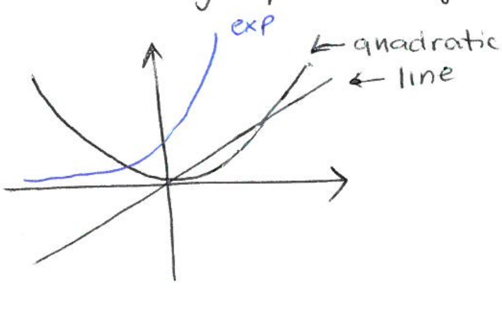
ie: $b^x > 0$ for all x. ie: exp is always pos

- when $b>1$ → increases rapidly ; $0<b<1$ → decreases rapidly





- exponentials grow (+ decay) rapidly
  - → faster than lines, quadratics etc.
  - faster than any power of x.



Properties of Exponentials

eg: $y = 2^x$

x	2^x	
1	2	$\times 2$
2	4	$\times 2$
3	8	$\times 2$
4	16	

Notice we are multiplying by a constant amount

We can recognise an exponential by this property.

Doubling Time:

Notice

x	2^x	
1	2	doubles
2	4	doubles.
3	8	

The "time" it takes to double in value is constant.

Here This function doubles its value every 1 sec

ie: The doubling time is constant

Also the tripling time is constant

And " quadrupling " " "

Quadrupling Time

x	2^x
1	2
2	4
3	8
4	16
5	32

$\times 4$

$\times 4$

Here the quadrupling time is 2 secs.

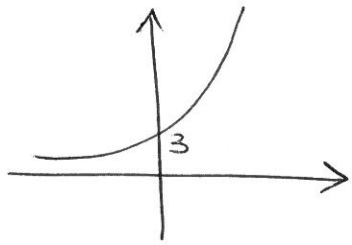
For any exponential : The time for it to be multiplied by a constant is constant.

In particular the halving time is constant (for exp decay)

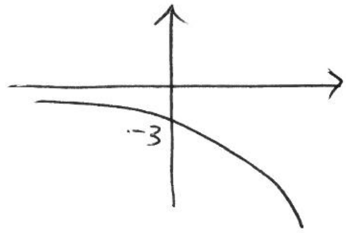
This is called the Half Life

The Exponential equation can have multiples out the front.

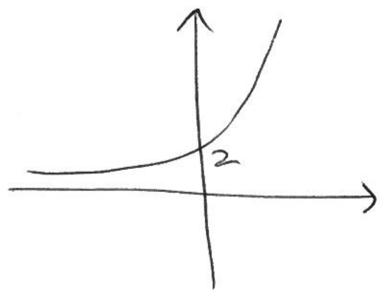
eg: $y = 3(2^x)$ is an exponential (all y-values are mult by 3)



$y = (-3)\,2^x$ looks like

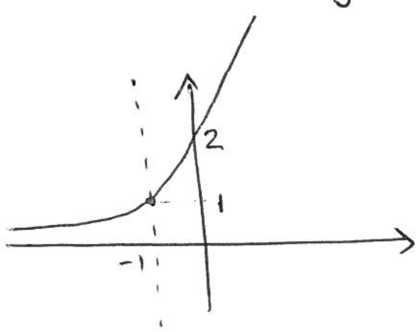


$y = 2 \cdot 2^x$



(y values doubled)

This is the same as $y = 2^{x+1}$ ← horizontal shift

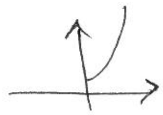


- The general equation of an exponential is of the form

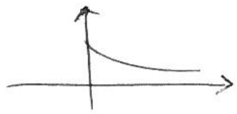
$$y = ab^x \qquad (b > 0,\ b \neq 1)$$

- Exponential models are used for particular growth + decay situations.

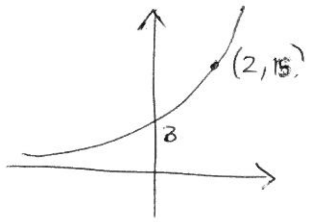
  eg: Population growth.



  - Radioactive decay



Eg Find the equation of the exponential which has a y-intercept of 3 and goes through (2,15)



Let $y = ab^x$

$(0,3)$: $3 = ab^0 \rightarrow a = 3$

$(2,15)$: $15 = ab^2$

$\therefore$ $15 = 3b^2$

$b^2 = 5$

$b = \pm\sqrt{5}$

Since $b > 0$, ignore neg

$\therefore$ $y = 3(\sqrt{5}^x)$

$= 3 \cdot 5^{x/2}$

Find the equation of the exponential which goes through the point (1, 4.8) and has a doubling time of 3 secs.

So $x = 1$, $y = 4.8$
when $x = 4$, $y = 9.6$ doubles.

$\therefore$ Curve goes through $(1, 4.8)$ and $(4, 9.6)$

Let $y = ab^x$

$(1, 4.8)$: $4.8 = ab$ — ①

$(4, 9.6)$: $9.6 = ab^4$ — ②

from ① : $a = \frac{4.8}{b}$

sub in ② : $9.6 = \left(\frac{4.8}{b}\right) \cdot b^4$

ie: $\frac{9.6}{4.8} = b^3$

ie: $b^3 = 2$

$\therefore$ $b = \sqrt[3]{2} = 2^{1/3}$

sub back in : $a = \frac{4.8}{2^{1/3}}$

$= 3.81$ (2dp)

$\therefore$ $y = (3.81)(2^{1/3})^x$

$= (3.81) \cdot 2^{x/3}$

The number of bacteria in a culture is given by the formula $Q(t) = 250 \cdot 3^{t/4}$ where t is measured in days. Estimate

a) the initial population

b) the population after 4 days

c) the population after 14 days.

a) Initially $\rightarrow t = 0$: $Q(t) = 250 \cdot 3^{(0)}$
$= 250$.

b) After 4 days $\rightarrow t = 4$: $Q = 250 \cdot 3^{4/4}$
$= 250 \times 3$
$= 750$

c) After 14 days $\rightarrow t = 14$: $Q = 250 \cdot 3^{14/4} = 250\,(3^{3.5})$
$= 11\,691$

Note: we can picture what's happening $Q(t) = 250 \cdot 3^{t/4}$
$\uparrow$
exp growth