Bit more on Compostion

$f(x) = x^2 + 1$ $\qquad$ $g(x) = \sqrt{x}$.

Few things to note:

$f(x) = x^2+1$ $\leftarrow$ we say this is "f of x"

NOT $f \times x$. (not mult)

+ we are apply the function f to the input x.

So $(f \circ g)(x) = f(g(x))$ is f of g

NOT $f \times g$.

Important that we work from inside out
+ Don't multiply.

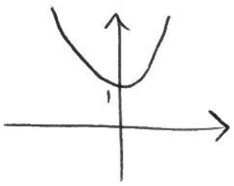
So $f(g(x)) = f(\sqrt{x})$

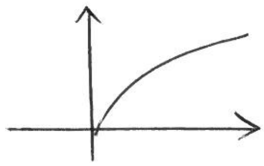
$= (\sqrt{x})^2 + 1$

$= x + 1$

Be aware of domains when doing composition

$f(x) = x^2 + 1$ $\qquad\qquad$ $g(x) = \sqrt{x}$





Dom $= \mathbb{R}$
Range $= [1, \infty)$

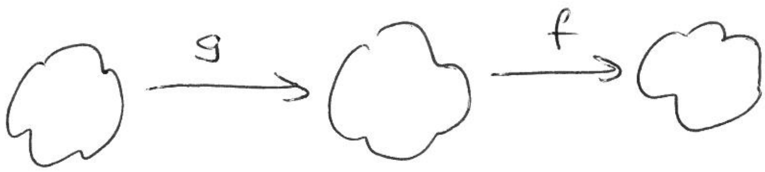
Dom $= [0, \infty)$
Range $= [0, \infty)$

We just saw $(f \circ g)(x) = \cdots = x + 1$

This new function looks like it has Dom $= \mathbb{R}$.

But Lets check:

$f(g(x))$:



Dom $= [0, \infty) \longrightarrow [0, \infty) \longrightarrow [1, \infty)$

applying g first so start with domain of g.

$\therefore$ Dom of $f \circ g(x) = x + 1$ is $[0, \infty)$

Eg5) Write $f(x)$ as the composition of 2 simpler functions.

a) $f(x) = (4x+3)^3$

Notice $4x+3$ is inside the cubed function

$\therefore$ Let $g(x) = 4x+3$ + $h(x) = x^3$

so $f(x) = h(g(x))$

b) $f(x) = \sqrt{x-1}$

$\therefore$ Let $g(x) = x-1$ + $h(x) = \sqrt{x}$

so $f(x) = h(g(x))$

c) $f(x) = \dfrac{1}{x^2-2}$

Let $g(x) = x^2-2$ and $h(x) = \dfrac{1}{x}$

so $f(x) = h(g(x))$

d) $f(x) = (5x+1)^2 - 2$

Take $g(x) = 5x+1$ and $h(x) = x^2-2$

So $f(x) = g(h(x))$

Inverses

- Start with some functions we know

$f(x)$	
$2x$	
$x+7$	
x^3	

- Find functions that undo them

$f(x)$	undo f
$2x$	$\frac{x}{2}$
$x+7$	$x-7$
x^3	$\sqrt[3]{x}$

- More formally

Let $f(x) = 2x$ $\quad g(x) = \frac{x}{2}$

So $x \xrightarrow{f} 2x \xrightarrow{g} x$

ie: $f \circ g(x) = f(g(x)) = f\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right) = x$

And $g \circ f(x) = g(f(x)) = g(2x) = \frac{2x}{2} = x$

We get back to what we started with

$f(x)$ and $g(x)$ undo each other

Similarly if $h(x) = x+7$ $\quad k(x) = x-7$

Then $h \circ k(x) = k \circ h(x) = x$

- Certain pairs of functions have the important property that they undo each other.

- We call functions that undo each other inverses.

- To represent this special relationship we have particular notation.

$f(x) = 2x$

we say $f^{-1}(x) = \frac{x}{2}$

↑

represents inverse of f.

(Note $f^{-1}(x) \neq \frac{1}{f(x)}$)

- So we say $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$

- Inverses are unique

- So $h(x) = x+7$ and $h^{-1}(x) = x-7$

Also $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$

We can show $h(x) = x+7$ and $h^{-1}(x) = x-7$ by looking at $h \circ h^{-1}$ and $h^{-1} \circ h$.

$$(h \circ h^{-1})(x) = h(h^{-1}(x))$$
$$= h(x-7)$$
$$= (x-7)+7$$
$$= x$$

$$(h^{-1} \circ h)(x) = h^{-1}(h(x))$$
$$= h^{-1}(x+7)$$
$$= (x+7)-7$$
$$= x$$

since $(h \circ h^{-1})(x) = x$ and $(h^{-1} \circ h)(x) = x$ these functions are inverses.

Finding Inverses

eg: $f(x) = 2x - 1$

ie: $y = 2x - 1$ $\qquad$ (y = output, x = input)

Inverse undoes this so outputs of inverse bring us back to inputs of original.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$\therefore$ inputs + outputs swap roles.

$\therefore$ For inverse fn we expect:

$$x = 2y - 1$$

making y the subject

$$y = x + 1$$

$$y = \frac{x+1}{2}$$

$$\therefore \quad f^{-1}(x) = \frac{x+1}{2}$$

eg. Find the inverse of $f(x) = \frac{x^3 - 1}{4}$

Let $y = \frac{x^3 - 1}{4}$

Swap $x = \frac{y^3 - 1}{4}$

Rearrange $y^3 - 1 = 4x$

$y^3 = 4x + 1$

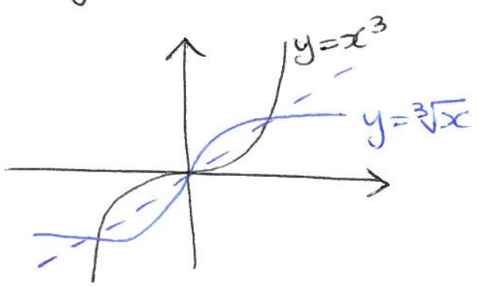
$y = \sqrt[3]{4x+1}$

$= (4x+1)^{1/3}$

$\therefore f^{-1}(x) = (4x+1)^{1/3}$

Special property of Inverses :

- They are symmetrical about the line $y = x$



- This is because of relationships between inputs + outputs (ie. dom + range)

Suppose $f: A \to B$

A = dom
B = range

Then $f^{-1}: B \to A$

Sometimes we have to be careful.

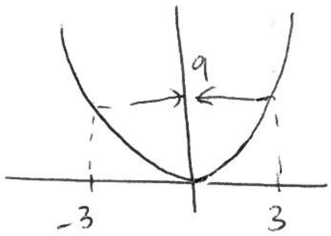
eg: $f(x) = x^2$

We know

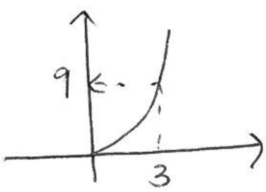
f	f^{-1}
x^2	$\sqrt{x}$

square + square root undo each other.

But notice



- Both -3 and $+3$ square to give 9
- To "undo" 9, which one do we go to?
  (can't go to both since then its not a function)
- Need 1 value to go to 1 value to get an inverse.
- Restrict Domain



Let $f: [0, \infty) \to [0, \infty)$ $f(x) = x^2$

and $f^{-1}(x) = \sqrt{x}$

- Algebraically $y = x^2$

  swap $x = y^2$

  rearrange $y = \pm\sqrt{x}$

  ↖ we chose $y = +\sqrt{x}$