③ Solve by completing the square.

eg 7a) Solve $x^2 - 2x - 6 = 0$

complete sq: $x^2 - 2x + 1 - 1 - 6 = 0$

$(x-1)^2 - 7 = 0$

$(x-1)^2 = 7$

$x - 1 = \pm\sqrt{7}$

$x = 1 \pm \sqrt{7}$

b) Solve $x^2 + 10x + 3 = 0$

complete sq: $x^2 + 10x + 25 - 25 + 3 = 0$

$(x+5)^2 - 22 = 0$

$(x+5)^2 = 22$

$x + 5 = \pm\sqrt{22}$

$x = 5 \pm \sqrt{22}$

④ Solve using the Quadratic Formula.

Recall the Quad formula: Solving $ax^2+bx+c=0$

gives us solutions $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

eg 8a) Solve $x^2+3x-5=0$.

$a=1 \quad b=3 \quad c=-5$

$$\therefore x=\frac{-3\pm\sqrt{3^2-4(-5)(1)}}{2(1)}$$

$$=\frac{-3\pm\sqrt{9+20}}{2}$$

$$=\frac{-3\pm\sqrt{29}}{2}$$

b) Solve $2x^2-4x+1=0$

$a=2 \quad b=-4 \quad c=1$

$$x=\frac{4\pm\sqrt{16-4(2)(1)}}{2(2)}$$

$$=\frac{4\pm\sqrt{8}}{4}$$

$$=\frac{4\pm2\sqrt{2}}{4}$$

$$=\frac{2(2\pm\sqrt{2})}{4}=\frac{2\pm\sqrt{2}}{2}$$

Proof of the Quadratic Formula

Solving $ax^2 + bx + c = 0$.

We will complete square to solve:

$$a\left[x^2 + \frac{b}{a}x\right] + c = 0$$

ie: $$a\left[\underbrace{x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2}_{} - \left(\frac{b}{2a}\right)^2\right] + c = 0$$

ie: $$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c = 0$$

ie: $$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

So $$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$= \frac{b^2 - 4ac}{4a}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

ie: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\blacksquare$

The Discriminant

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Recall b^2-4ac determine how many solutions we ha.

$\Delta = b^2 - 4ac$ = discriminant

If $\Delta > 0 \rightarrow$ 2 solutions

$\Delta < 0 \rightarrow$ No solution

$\Delta = 0 \rightarrow$ 1 solution

eg1a) $x^2 - 2x + 4 = 0$

$a=1 \quad b=-2 \quad c=4$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(4) \\ &= 4 - 16 \\ &< 0\end{aligned}$$

$\therefore$ No solutions

b) $4x^2 - 12x + 9 = 0$

$\Delta = (12)^2 - 4(4)(9) = 144 - 144 = 0$ $\therefore$ 1 soln

ie: $x = \dfrac{-12 \pm \sqrt{288}}{2(4)} = \dfrac{-12}{8} = -\dfrac{3}{2}$

c) $x^2 + 4x - 1 = 0$

$\Delta = (4)^2 - 4(1)(-1) = 16 + 4 = 20$ $\therefore$ 2 solns

$\therefore x = \dfrac{-4 \pm \sqrt{20}}{2} = \dfrac{-4 \pm 2\sqrt{5}}{2} = \dfrac{\cancel{2}(-2 \pm \sqrt{5})}{\cancel{2}} = 2 \pm \sqrt{5}$

Back to sketching parabolas

- Now that we know how to solve quadratic equations we can put more details into our sketches.

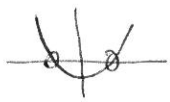
eg1a) Find the x intercepts and sketch $y = x^2 - 6x + 8$

a) $y = x^2 - 6x + 8$

↖ parabola facing up (concave up)

Recall: x intercepts = where graph cuts x-axis

→ ie: points when $y = 0$.

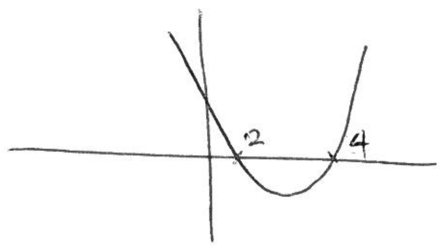


$\therefore$ To find x int → Let $y = 0$.

$\therefore$ $x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

$x = 2, 4$



- we found 2 x-intercepts
- we call these solutions <u>roots</u> of the quadratic

or <u>zeros</u>

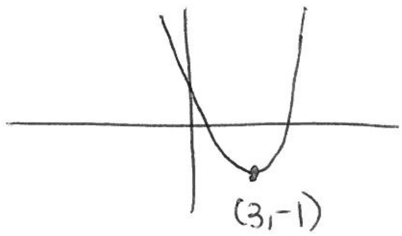
A few more details:

note we can verify our sketch by completing the square.
(Because of symmety we expect our vertex to occur when $x=3$).

b) $y = x^2 - 6x + 8$

$= x^2 - 6x + 9 - 9 + 8$

$= (x-3)^2 - 1$



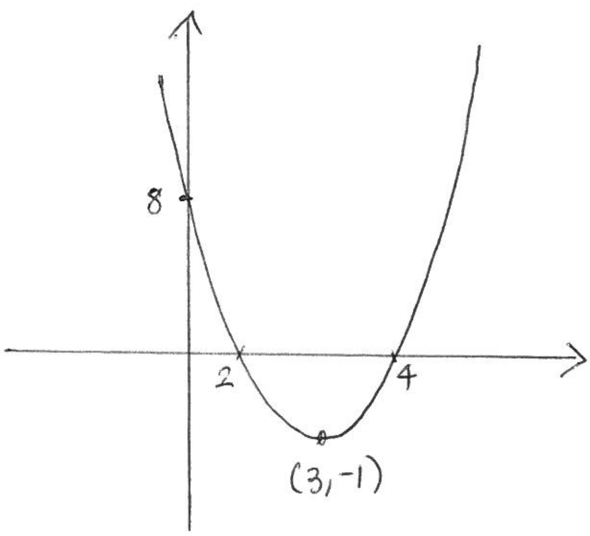
c) vertex $= (3,-1)$

d) Minimum value $= 3$
This occus when $x = -1$.

e) We can also find the y-intercept
→ Let $x = 0$

$\therefore\ y = 0^2 - 6(0) + 8 = 8$

f)



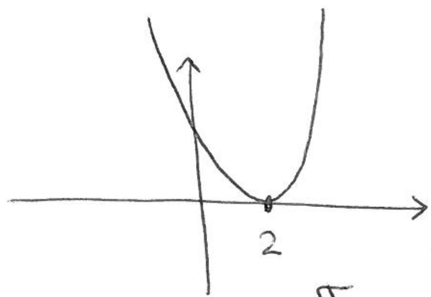
1a) $y = 3x^2 - 12x + 12$

Notice $b^2 - 4ac = (-12)^2 - 4(3)(12) = 0$

$\therefore$ There is 1 root at $x = \dfrac{12 \pm \sqrt{0}}{2(3)} = 2$

This parabola faces up.

$\therefore$ It must look like



This is a double root.

check: $y = 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

$= 3(x-2)^2$ ← horiz shift to right.

y int → Let $x = 0$: $y = 12$.

b) $y = -3 - 2x - x^2$

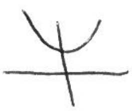
Notice $b^2 - 4ac = (-2)^2 - 4(-1)(-3) = 4 - 12 < 0$

$\therefore$ No solutions

$\therefore$ No roots

$\therefore$ No x-intercepts.

Parabbla still exists, it just doesnt cut x axis.

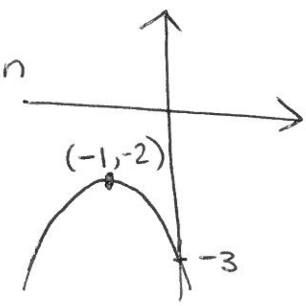
$\therefore$ Either  or 

- Since neg in front of x^2, must be concave down

- complete sq: $y = -(x^2 + 2x + 3)$

$= -(x^2 + 2x + 1 + 2)$

$= -(x+1)^2 - 2$



e) $y = x^2 - 2x - 1$

Finding x int : $b^2 - 4ac = (-2)^2 - 4(1)(-1)$

$= 8$

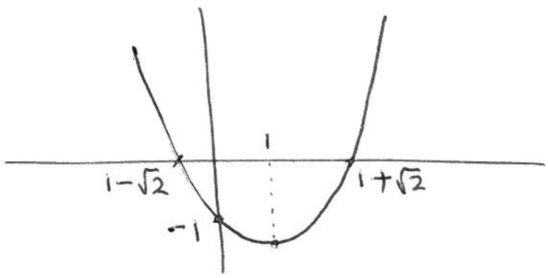
$\therefore$ using quad formula: $x = \frac{-(-2) \pm \sqrt{8}}{2}$

$= \frac{2 \pm 2\sqrt{2}}{2}$

$= \frac{\cancel{2}(1 \pm \sqrt{2})}{\cancel{2}}$

this parabola is concave up + y int is -1

$\therefore$



we can see vertex occurs when $x = 1$ (by symmetry)

$\therefore$ Find minimum by substituting:

when $x = 1$: $y = (1)^2 - 2(1) - 1$

$= -2$

$\therefore$ vertex is $(1, -2)$

Quadratic Models

- Quadratic equations model some real life problems (eg: projectile motion)
- We need to be able to work with quadratic eqns

Eg12) a) Solve $V = \pi r^2 h$ for r

$$V = \pi r^2 h \qquad (\text{Aim: } r = \ )$$

$$\frac{V}{\pi h} = r^2$$

$$r = \pm\sqrt{\frac{V}{\pi h}}$$

b) Solve $F = \dfrac{G m_1 m_2}{r^2}$ for r

$$\therefore F r^2 = G m_1 m_2$$

$$r^2 = \frac{G m_1 m_2}{F}$$

$$r = \pm\sqrt{\frac{G m_1 m_2}{F}}$$

c) Solve $A = 2x^2 + 4xh$ for x.

Aim $x =$

Assume A, h are constants.

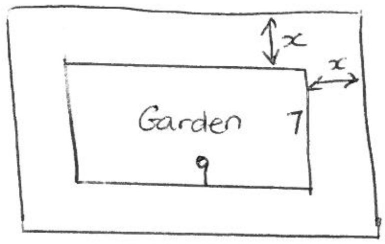
$\therefore$ $2x^2 + 4xh - A = 0$ is a quadratic in x

use quad formula $(a = 2,\ b = 4h,\ c = -A)$

$$x = \frac{-4h \pm \sqrt{16h^2 - 4(2)(-A)}}{2(2)}$$

$$= \frac{-4h \pm \sqrt{16h^2 + 8A}}{4}$$

. A rectangular garden is 9m by 7m. It is planned to construct a path of uniform width, x metres, around the garden. There is enough concrete so that the area taken by the path is 50 square metres. Find the width of the path.



. Want to find x

- We know . Area of path = 50

. Area of garden = 63

$\therefore$ Area of path = Area of Big Rectangle - Area of garden

$= (2x+7)(2x+9) - 63$

$\therefore \quad (2x+7)(2x+9) - 63 = 50$

$4x^2 + 18x + 14x + 63 - 63 = 50$

$4x^2 + 32x - 50 = 0$

$2x^2 + 16x - 25 = 0$

$\therefore \quad x = \dfrac{-16 \pm \sqrt{(16)^2 - 4(2)(-25)}}{2(2)}$

$= \dfrac{-16 \pm \sqrt{256 + 200}}{4}$

$x = \dfrac{-16 + \sqrt{456}}{4}, \quad x = \dfrac{-16 - \sqrt{456}}{4}$

$= 1.34$ $\leftarrow$ Neg $\therefore$ Ignore

$\therefore$ Width of path is 1.34m.