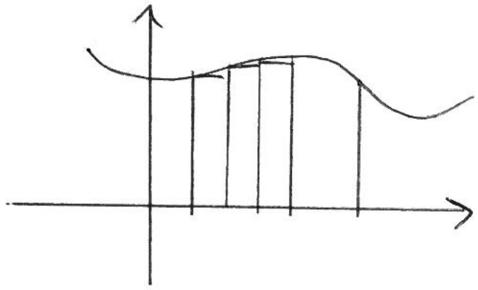
Numerical Integration

- In real life we often dont get a nice function we know how to integrate ∴ our estimations of areas under a curve are important.

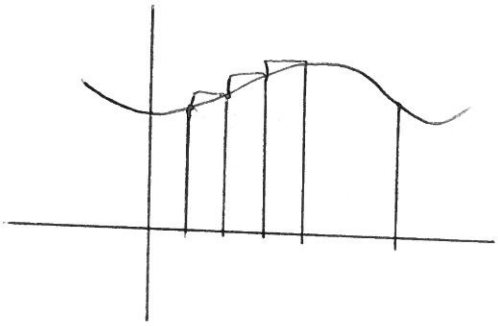
- We already saw some ways of approximating the area under a curve.

· Left-end rule



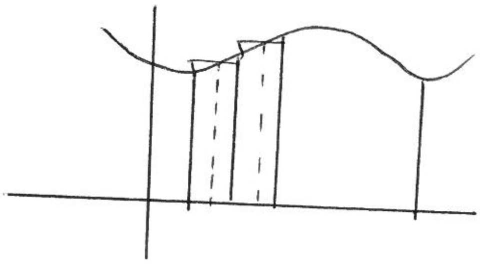
use the left point of subinterval for height.

· Right-end rule



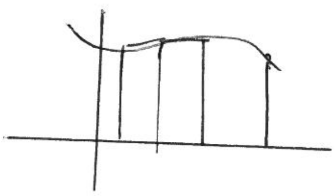
use the right point of subinterval for height

· Midpoint rule



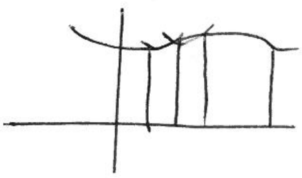
use the midpt of subinterval for height.

· Trapezoidal rule



form trapeziums

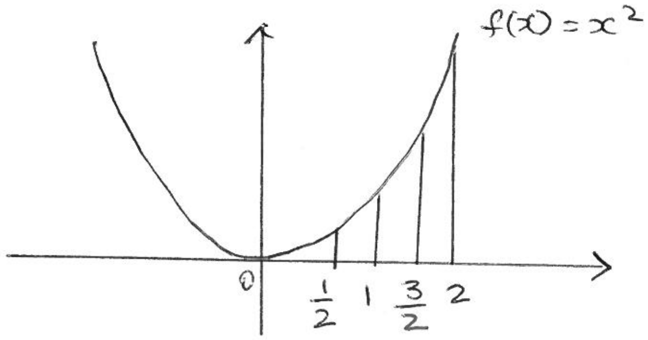
· Simpsons rule



fit parabolas

- we will concentrate on the first 3 of these
  - Left end
  - Right end
  - Mid point.

eg: Use the left-end, right-end + midpoint rules to find $\int_0^2 x^2\, dx$ using 4 sub-intervals.



width $= \frac{2-0}{4} = \frac{1}{2}$

Left end : $L_4 = \frac{1}{2} f(0) + \frac{1}{2} f(\frac{1}{2}) + \frac{1}{2} f(1) + \frac{1}{2} f(\frac{3}{2})$

$= \frac{1}{2}\left[0^2 + (\frac{1}{2})^2 + (1)^2 + (\frac{3}{2})^2\right]$

$= \frac{1}{2}\left[\frac{1}{4} + 1 + \frac{9}{4}\right]$

$= \frac{1}{2}\left(\frac{14}{4}\right)$

$= \frac{7}{4}$ $(= 1.75)$

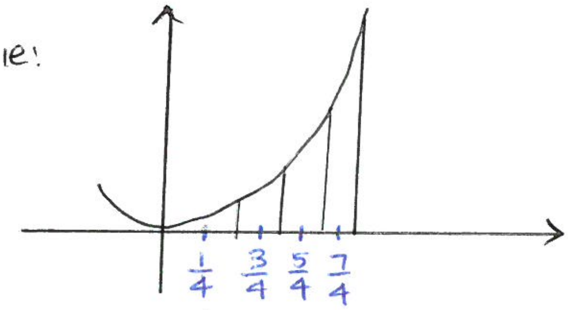
Right end : $R_4 = \frac{1}{2}\left[f(\frac{1}{2}) + f(1) + f(\frac{3}{2}) + f(2)\right]$

$= \frac{1}{2}\left[(\frac{1}{2})^2 + 1^2 + (\frac{3}{2})^2 + 2^2\right]$

$= \frac{15}{4}$ $(= 3.75)$

midpoint → Now use midpoint for height

ie:



$$M_4 = \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{5}{4}\right)^2 + \left(\frac{7}{4}\right)^2\right)$$

$$= \frac{85}{32} \quad (= 2.656..)$$

Note: The true area $= \int_0^2 x^2\,dx$

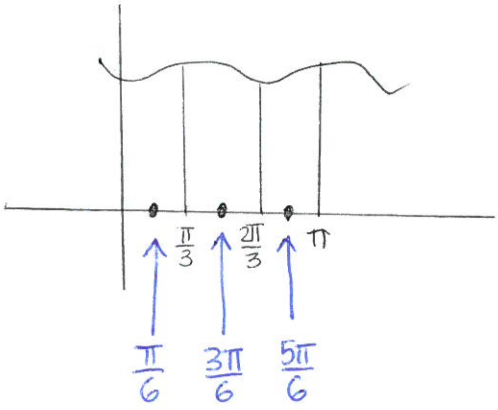
$$= \left.\frac{x^3}{3}\right|_0^2$$

$$= \frac{2^3}{3} - 0$$

$$= \frac{8}{3}$$

$$= 2.6667$$

Eg: Use the mid point rule to approximate $\int_0^{\pi} \sqrt{\sin x}\, dx$ using 3 sub-intervals.



$f(x) = \sqrt{\sin x}$

width $= \frac{\pi - 0}{3} = \frac{\pi}{3}$

Area $= \sum$ width $\times$ height

$= \sum \frac{\pi}{3} \times$ function value at each point

$= \frac{\pi}{3}\left(f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{5\pi}{6}\right)\right)$ $\leftarrow$ so $f\left(\frac{\pi}{6}\right) = \sqrt{\sin \frac{\pi}{6}}$ $= \sqrt{1/2}$

$= \frac{\pi}{3}\left(\sqrt{\frac{1}{2}} + \sqrt{1} + \sqrt{1/2}\right)$

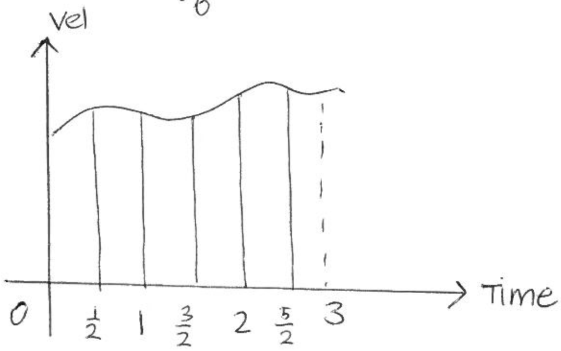
$= \frac{\pi}{3}\left(1 + 2\sqrt{1/2}\right)$

$= 2.528\ldots$

Note: The more subintervals we take, the more accurate the estimate for the area.

eg: The velocity of an insect is given by $v(t) = \ln(t^2+1)$ m/s for $0 \le t \le 3$ where t is in seconds. Find the distance the insect has travelled during this time using the left-end approximation with 6 sub divisions.

ie: We want $\int_0^3 \ln(t^2+1)\,dt$ ← distance = area under curve.



(this is actually displacement by $v(t)$ sits above x-axis so it's also distance).

$$\text{Width} = \frac{3-0}{6} = 0.5$$

$$f(x) = \ln(t^2+1)$$

$$\text{Dist} = \text{Area} = \sum \text{width} \times \text{height}$$

$$= 0.5\left(f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)\right)$$

$$= 0.5\left(0 + \ln(1.25) + \ln 2 + \ln(3.25) + \ln 5 + \ln 7.25\right)$$

$$= 2.84 \text{ m}$$

Revision

Question 8 [15 marks]

a) Evaluate each of the following integrals:

(i) $\displaystyle\int \left(3x^2 + 4x^{-5/2} - 2\sqrt{x} + 2\right) \mathrm{d}x$ **(ii)** $\displaystyle\int \frac{x}{\sqrt{1-x^2}} \mathrm{d}x$

(iii) $\displaystyle\int_{\pi/6}^{\pi/4} \sin 2x \, \mathrm{d}x$ **(iv)** $\displaystyle\int_0^1 3x\mathrm{e}^{x^2} \mathrm{d}x$

b) The graphs of $y = 4 - x^2$ and $y = x + 2$ intersect at two points. Find the points, and find the area of the region bounded by the two graphs.

c) The velocity of a car, in metres per second, between the time the brakes are applied and the time it comes to a stop, is given by $v = 20 - 5t$, where t is the time in seconds after the brakes are applied.

(i) How long does the car take to stop?

(ii) How far has it travelled in that time?

Eg: Past paper Question

a) (i) $\int 3x^2 + 4x^{-5/2} - 2\sqrt{x} + 2 \, dx$

$= \int 3x^2 + 4x^{-5/2} - 2x^{1/2} + 2 \, dx$

$= \frac{3x^3}{3} + 4\frac{x^{-3/2}}{-3/2} - 2\frac{x^{3/2}}{3/2} + 2x + c$

$= x^3 - \frac{8}{3}x^{-3/2} - \frac{4}{3}x^{3/2} + 2x + c$

(ii) $\int \frac{x}{\sqrt{1-x^2}} \, dx$

let $u = 1 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$= \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{-2\cancel{x}}$

$= -\frac{1}{2}\int u^{-1/2} \, du$

$= -\frac{1}{2} \frac{u^{1/2}}{1/2} + c$

$= -u^{1/2} + c$

$= -\sqrt{1-x^2} + c$

(iii) $\int_{\pi/6}^{\pi/4} \sin 2x \, dx$

$= \left[\frac{-1}{2} \cos 2x \right]_{\pi/6}^{\pi/4}$

$= -\frac{1}{2}\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{3} \right)$

$= \frac{-1}{2}\left(0 - \frac{1}{2} \right)$

$= \frac{1}{4}$

(IV) $\displaystyle\int_0^1 3xe^{x^2}\,dx$

Let $u = x^2$

$\dfrac{du}{dx} = 2x$

$dx = \dfrac{du}{2x}$

when $x=0 \to u=0$

$x=1 \to u=1$

$= \displaystyle\int_0^1 3\cancel{x}\cdot e^u \cdot \frac{du}{2\cancel{x}}$

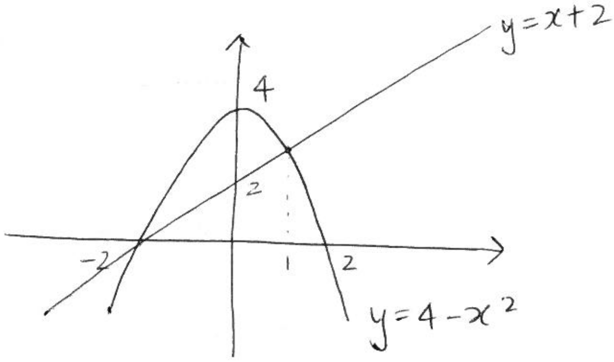
$= \dfrac{3}{2}\displaystyle\int_0^1 e^u\,du$

$= \dfrac{3}{2}\left[e^u\right]_0^1$

$= \dfrac{3}{2}\left[e^1 - e^0\right]$

$= \dfrac{3}{2}(e-1)$

b)



Finding pt of int: $4 - x^2 = x + 2$

ie: $x^2 + x - 2 = 0$

ie: $(x+2)(x-1) = 0$

$x = -2, 1$

Area $= \displaystyle\int_{-2}^{1} 4 - x^2 - (x+2)\,dx$

$= \displaystyle\int_{-2}^{1} 2 - x^2 - x\,dx$

$= \left[2x - \dfrac{x^3}{3} - \dfrac{x^2}{2}\right]_{-2}^{1}$

$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - \frac{4}{2}\right) = \dfrac{9}{2}$

c) $v = 20 - 5t$

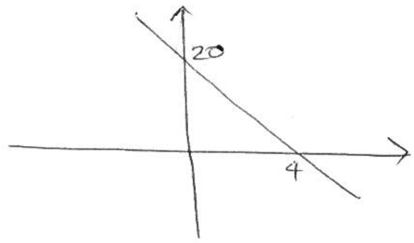
(i) Car will stop when $v=0$

ie: want t when $v=0$

ie: $20 - 5t = 0$

$5t = 20$

$t = 4$



$\therefore$ It takes 4 secs to stop.

(ii) want dist travelled in 4 secs.

$\therefore \quad s = \int_0^4 v \, dt$

$= \int_0^4 20 - 5t$

$= \left[20t - \frac{5t^2}{2} \right]_0^4$

$= \left(80 - \frac{5(4)^2}{2} \right) - 0$

$= 40 \text{ m}$