Quick Refresher (ques from past papers)

Eg 1) $\int \frac{3x^3}{\sqrt{x^4-1}} \, dx$

Let $u = x^4 - 1$

$\frac{du}{dx} = 4x^3$

$dx = \frac{du}{4x^3}$

$= \int \frac{3x^3}{\sqrt{u}} \cdot \frac{du}{4x^3}$

$= \frac{3}{4} \int u^{-1/2} \, du$

$= \frac{3}{4} \frac{u^{1/2}}{1/2} + c$

$= \frac{6}{4} u^{1/2} + c$

$= \frac{3}{2} \sqrt{x^4 - 1} + c$

2) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

Let $u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2} x^{-1/2}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x} \, du$

$= 2u \, du$

$= \int \frac{e^u}{\cancel{u}} \cdot 2\cancel{u} \, du$

$= 2e^u + c$

$= 2e^{\sqrt{x}} + c$

Definite Integrals + Substitution

eg) $\displaystyle\int_0^2 \frac{3x}{x^2+1}\,dx$

Let $u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

when $x=0 \rightarrow u=1$

$x=2 \rightarrow u=5$

$= \displaystyle\int_1^5 \frac{3x}{u} \cdot \frac{du}{2x}$

$= \displaystyle\frac{3}{2}\int_1^5 \frac{1}{u}\,du$

$= \frac{3}{2}\left[\log u\right]_1^5$

$= \frac{3}{2}\left(\log 5 - \log 1\right)$

$= \frac{3}{2}\log 5$

OR Method 2 : $\displaystyle\int_0^2 \frac{3x}{x^2+1}\,dx = \ldots = \frac{3}{2}\int_0^2 \frac{1}{u}\,du$

$= \frac{3}{2}\left[\log u\right]_0^2$

$= \frac{3}{2}\left[\log(x^2+1)\right]_0^2$

$= \frac{3}{2}\left(\log 5 - \log 1\right)$

$= \frac{3}{2}\log 5.$

b) $\int_0^1 x^2(1-2x^3)^3 \, dx$

Let $u = 1-2x^3$

$\frac{du}{dx} = -6x^2$

$dx = \frac{du}{-6x^2}$

$= \int_1^{-1} x^2 . u^3 . \frac{du}{-6x^2}$

when $x=0 \rightarrow u=1$

$x=1 \rightarrow u=1-2=-1$

$= -\frac{1}{6}\int_1^{-1} u^3 \, du$

$= \frac{1}{6}\int_{-1}^{1} u^3 \, du$ $\longleftarrow$ Remember $\int_a^b = -\int_b^a$

$= \frac{1}{6}\left[\frac{u^4}{4}\right]_{-1}^{1}$

$= \frac{1}{6}\left(\frac{1}{4} - \frac{(-1)^4}{4}\right)$

$= 0$

c) $\int_0^4 \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

Let $u = \sqrt{x} = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x} \, du = 2u\,du$

$= \int_0^2 \frac{\sin u}{u} 2u . du$

when $x=0 \rightarrow u=\sqrt{0}=0$

$x=4 \rightarrow u=\sqrt{4}=2$

$= 2\int_0^2 \sin u \, du$

$= 2\left[-\cos u\right]_0^2$

$= 2(-\cos 2 - (-\cos 0))$

$= 2(-\cos 2 + 1)$

$= 2(1-\cos 2)$

A few more involving trig

a) $\int e^{\sin x} . \cos x \; dx$

Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \int e^u . \cancel{\cos x} \frac{du}{\cancel{\cos x}}$

$= \int e^u \; du$

$= e^u + c$

$= e^{\sin x} + c$

← Another rule to notice

$\int f'(x) . e^{f(x)} \; dx = e^{f(x)} + c$

b) $\int \tan x \; dx = \int \frac{\sin x}{\cos x} \; dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{du}{-\sin x}$

$= \int \frac{\cancel{\sin x}}{u} . \frac{du}{-\cancel{\sin x}}$

$= -\int \frac{1}{u} \; du$

$= -\log|u| + c$

$= -\log|\cos x| + c$

c) $\int \sin x \cdot \cos x \; dx$

Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$$= \int u \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int u \cdot du$$

$$= \frac{u^2}{2} + c$$

$$= \frac{\sin^2 x}{2} + c$$

Method 2: $\int \sin x \cos x \; dx$

$$= \frac{1}{2} \int \sin 2x \; dx$$

Remember

$\sin 2x = 2 \sin x \cos x$

$\therefore \frac{1}{2} \sin 2x = \sin x \cos x$

$$= \frac{1}{2} \left[-\frac{1}{2} \cos 2x \right] + c_1$$

$$= -\frac{1}{4} \cos 2x + c_1$$

By why are these answers different?

$$-\frac{1}{4} \cos 2x = -\frac{1}{4} (\cos^2 x - \sin^2 x) \quad + c$$

$$= -\frac{1}{4} (1 - \sin^2 x - \sin^2 x) \quad ''$$

$$= -\frac{1}{4} (1 - 2\sin^2 x) \quad ''$$

$$= -\frac{1}{4} + \frac{1}{2} \sin^2 x \quad ''$$

$$= \frac{1}{2} \sin^2 x - \frac{1}{4} \quad ''$$

So this $-1/4$ is explained by constant.

ie: $C_1 - C = 1/4$

d) $\int \cos x \cdot \sin^3 x \; dx$

Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \int \cos x \cdot u^3 \cdot \frac{du}{\cos x}$

$= \int u^3 \; du$

$= \frac{u^4}{4} + c$

$= \frac{\sin^4 x}{4} + c$

e) $\int \sin^3 x \; dx$

$= \int \sin x \cdot \sin^2 x \; dx$

$= \int \sin x \; (1 - \cos^2 x) \; dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{-du}{\sin x}$

$= \int \cancel{\sin x} \; (1 - u^2) \frac{-du}{\cancel{\sin x}}$

$= -\int 1 - u^2 \; du$

$= \int u^2 - 1 \; du$

$= \frac{u^3}{3} - u + c$

$= \frac{\cos^3 x}{3} - \cos x + c$

f) $\int \frac{\cos x}{(1+\sin x)^2}\,dx$

Let $u = 1+\sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \int \frac{\cancel{\cos x}}{u^2} \cdot \frac{du}{\cancel{\cos x}}$

$= \int u^{-2}\,du$

$= \frac{u^{-1}}{-1} + C$

$= \frac{-1}{u} + C$

$= \frac{-1}{1+\sin x} + C$

There are many more techniques of integration but often we end up with functions that we don't know how to integrate.

$\therefore$ We rely on our estimation methods.