Integration Techniques

Revision

Rules:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int k \, dx = kx + c \qquad (k = \text{constant})$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \log|x| + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

Revision - examples from past papers:

i) $\int x^4 - \frac{1}{x^3} + \frac{2}{x} + \sin 2x + e^{4x} \, dx$

$$= \int x^4 - x^{-3} + 2\left(\frac{1}{x}\right) + \sin 2x + e^{4x} \, dx$$

$$= \frac{x^5}{5} - \frac{x^{-2}}{-2} + 2\log|x| + \frac{1}{2}(-\cos 2x) + \frac{1}{4}e^{4x} + c$$

$$= \frac{x^5}{5} - \frac{1}{2x^2} + 2\log|x| - \frac{1}{2}\cos 2x + \frac{1}{4}e^{4x} + c$$

2) $\int e^{7-6x}\,dx = \frac{-1}{6}e^{7-6x} + c$

3) $\int \frac{1}{3x-7}\,dx = \frac{1}{3}\log|3x-7| + c$

4) $\int \frac{1}{(3x-7)^2}\,dx = \int (3x-7)^{-2}\,dx$

$= \frac{1}{3}\frac{(3x-7)^{-1}}{-1} + c$

$= \frac{-1}{3}\cdot(3x-7)^{-1} + c$

$= \frac{-1}{3(3x-7)} + c$

Techniques of Integration

Integration by Substitution

= method of integrating to "change" the integral into something we can do.

eg) $\int \frac{x}{\sqrt{1+x^2}}\,dx$

Let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \int \frac{\cancel{x}}{\sqrt{u}} \cdot \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\left[2u^{1/2}\right] + c$

$= u^{1/2} + c$

$= \sqrt{1+x^2} + c$

Notice:
- This worked because we could cancel out the x's
- The x is basically like the derivative of $1+x^2$ (just out by a constant)
- This is our clue that the method of substitution could work

ie! Look out for a function and its derivative

Eg. a) $\int x^2(x^3-2)^5 \, dx$

Let $u = x^3 - 2$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$= \int \cancel{x^2} \cdot u^5 \cdot \frac{du}{3\cancel{x^2}}$

$= \frac{1}{3}\int u^5 \, du$

$= \frac{1}{3}\left[\frac{u^6}{6}\right] + c$

$= \frac{u^6}{18} + c$

$= \frac{(x^3-2)}{18} + c$

b) $\int x^2 e^{x^3} \, dx$

Let $u = x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$= \int \cancel{x^2} e^u \frac{du}{3\cancel{x^2}}$

$= \frac{1}{3}\int e^u \, du$

$= \frac{1}{3}e^u + c$

$= \frac{1}{3}e^{x^3} + c$

c) $\int x^4 \cos(x^5+3)\, dx$

Let $u = x^5+3$

$\frac{du}{dx} = 5x^4$

$dx = \frac{du}{5x^4}$

$= \int \cancel{x^4} \cos u \cdot \frac{du}{5\cancel{x^4}}$

$= \frac{1}{5}\int \cos u \, du$

$= \frac{1}{5} \sin u + C$

$= \frac{1}{5} \sin(x^5+3) + c$

d) $\int \frac{x}{x^2+1}\, dx$

Let $u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$

$= \frac{1}{2}\int \frac{1}{u} \cdot du$

$= \frac{1}{2} \log u + c$

$= \frac{1}{2} \log(x^2+1) + c$

e) $\int \frac{2x}{x^2+4}\, dx$

Let $u = x^2+4$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$= \int \frac{2\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$

$= \log(x^2+4) + c$

← Notice Derivative of Bottom is on top

f) $\int \frac{3x^2}{5+x^3}\,dx$ ← derivative of bottom is on the top.

$= \int \frac{\cancel{3x^2}}{u} \cdot \frac{du}{\cancel{3x^2}}$

ie: $u = 5+x^3$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

$= \int \frac{1}{u}\,du$

$= \log u + c$

$= \log(5+x^3) + c$

← Look out for $\frac{f'(x)}{f(x)}$

ie: $\boxed{\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c}$

g) $\int \frac{2x+1}{x^2+x-3}\,dx$ ← Notice $\int \frac{f'}{f}$

$= \log(x^2+x-3) + c$

h) $\int \frac{6x^2+8x}{x^3+2x^2+7}\,dx$ ← Notice the deriv of bottom is $3x^2+4x$

$= \int \frac{2(3x^2+4x)}{x^3+2x^2+7}\,dx$

$= 2\log(x^3+2x^2+7) + C$

i) $\int \frac{(\log x)^2}{x}\,dx$ Let $u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$= \int \frac{u^2}{x} \cdot x\,du$

$= \frac{u^3}{3} + C$

$= \frac{(\log x)^3}{3} + C$

j) $\int \sin(3x+2)\,dx$ Let $u = 3x+2$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$= \int \sin u \cdot \frac{du}{3}$ ← We could do this more quickly using the reverse chain rule.

$= -\frac{1}{3}\cos u + C$

$= -\frac{1}{3}\cos(3x+2) + C$

Note that our reverse chain rules can also be done using substitution.

ie: $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

⋮

etc

$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

⋮

etc