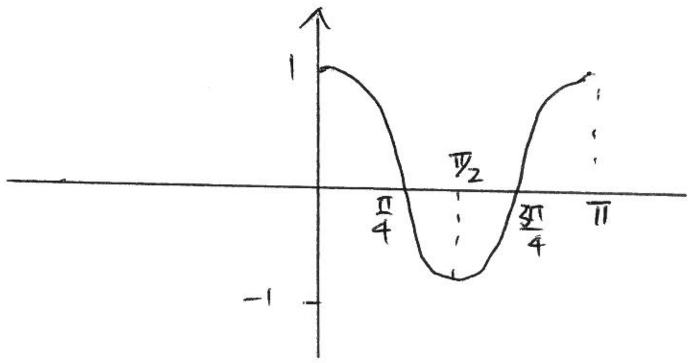
eg: Consider $f(x) = \cos 2x$.

a) Find the area between $y=f(x)$ + the x axis between $x=0$ and $x=\pi/4$.

Draw



$f(x) = \cos 2x$

$\uparrow$

period $= \frac{2\pi}{2} = \pi$

$$\text{Area} = \int_0^{\pi/4} \cos 2x \, dx$$

$$= \left[\tfrac{1}{2} \sin 2x \right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left(\sin \tfrac{\pi}{2} - \sin 0\right)$$

$$= \tfrac{1}{2}(1-0)$$

$$= \tfrac{1}{2} \text{ sq units}$$

b) Find the area between $y=f(x)$ + the x-axis between $x=0$ and $x=\pi/2$

By symmerty Area $= 2 \times \int_0^{\pi/4} \cos 2x \, dx$

$$= 2\left(\tfrac{1}{2}\right)$$

$$= 1$$

c) Find the area between $y=f(x)$ + the x axis between $x=-\pi/2$ and $x=\pi/2$.

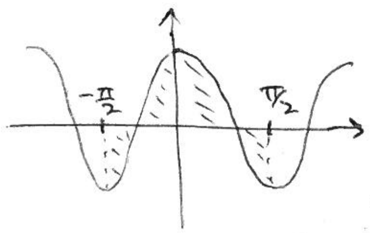
$$\text{Area} = 4 \times \int_0^{\pi/4} \cos 2x \, dx$$

$$= 4\left(\tfrac{1}{2}\right)$$

$$= 2 \text{ sq units}$$

d) Find the integral $\int_{-\pi/2}^{\pi/2} \cos 2x \, dx$

We can see by symmetry

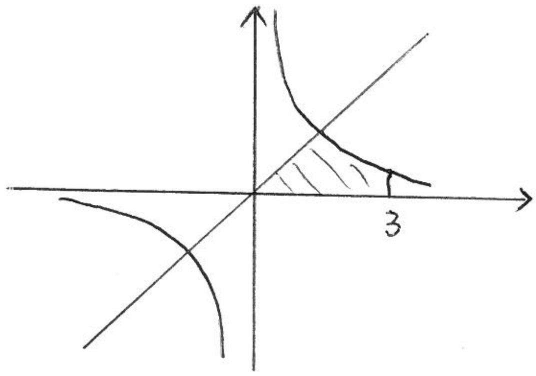


$$\int_{-\pi/2}^{\pi/2} \cos 2x \, dx = 0$$

Other reasons to split integrals :

Eg: Find the area enclosed by $y = x$, $y = \frac{1}{x}$ and the x-axis for $0 \leq x \leq 3$.

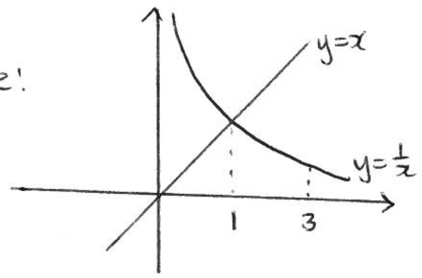
Draw :



Find pt of intersection:

$$x = \frac{1}{x}$$
$$x^2 = 1$$
$$x = \pm 1$$

ie:



$$\text{Area} = \int_0^1 x \, dx + \int_1^3 \frac{1}{x} \, dx$$

$$= \left[\frac{x^2}{2} \right]_0^1 + \left[\log x \right]_1^3$$

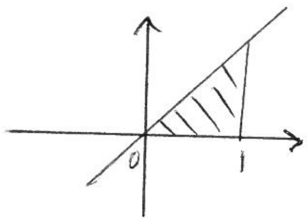
$$= \left(\frac{1}{2} - 0 \right) + \left(\log 3 - \log 1 \right)$$

$$= \frac{1}{2} + \log 3 \quad \text{sq units}$$

Looking at areas geometrically

Sometimes this is quicker (and easier)

eg) $\int_0^1 x \, dx$



Area of a triangle $= \frac{1}{2} bh$

$= \frac{1}{2} 1 \cdot 1$

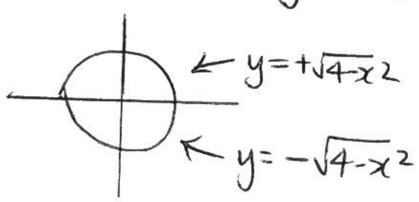
$= \frac{1}{2}$

2) $\int_{-2}^{2} \sqrt{4-x^2} \, dx$

Note circle: $x^2 + y^2 = 4$

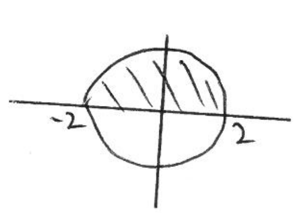
$y^2 = 4 - x^2$

$y = \pm \sqrt{4-x^2}$

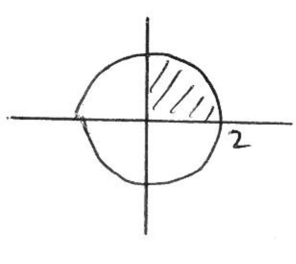


$\therefore \int_{-2}^{2} \sqrt{4-x^2} \, dx$ ← top half of semicircle

+ Area $= \frac{1}{2}\pi(2)^2 = 2\pi$



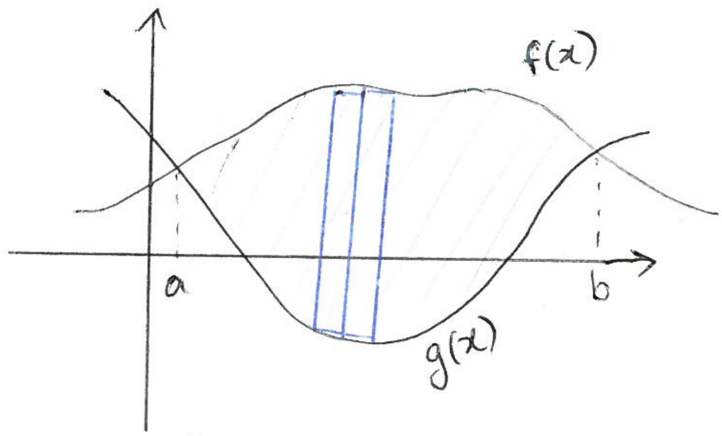
3) $\int_0^2 \sqrt{4-x^2} \, dx$



$= \frac{1}{4}\pi(2)^2$

$= \pi$

Area between two curves



Area = sum of rectangles

$= \sum_i$ width × height

$= \sum_i \Delta x \times$ (top value − bottom value)

$= \sum_i$ (top − bottom) Δx

$= \sum_i \left(f(x_i) - g(x_i) \right) \Delta x$

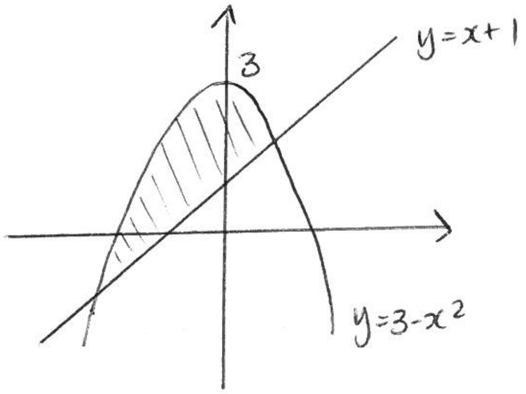
Let $\Delta x \to 0$: Area $= \int_a^b f(x) - g(x) \, dx$

OR Area $= \int_a^b$ top − bottom dx

Note : Now it doesn't matter where the curve lies on the plane − no need to worry about negative areas below axis.

Eg: Find the area between the two curves $y = 3 - x^2$ and $y = x + 1$.

Draw:



Find pts of int: $3 - x^2 = x + 1$

ie: $x^2 + x - 2 = 0$

ie: $(x+2)(x-1) = 0$

$x = -2, 1$

$\therefore$ Area $= \displaystyle\int_{-2}^{1} \text{top} - \text{bottom} \, dx$

$$= \int_{-2}^{1} 3 - x^2 - (x+1) \, dx$$

$$= \int_{-2}^{1} 2 - x^2 - x \, dx$$

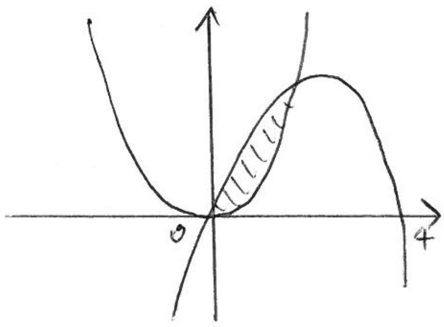
$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_{-2}^{1}$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right)$$

$$= \frac{9}{2} \text{ sq units}$$

Eg) Find the area between the parabolas $y = x^2$ and $y = x(4-x)$.

Draw:



Find pt of int:

$$x^2 = x(4-x)$$
$$x^2 = 4x - x^2$$
$$2x^2 - 4x = 0$$
$$2x(x-2) = 0$$
$$x = 0, 2$$

$$\text{Area} = \int_0^2 \text{top} - \text{bottom} \, dx$$

$$= \int_0^2 x(4-x) - x^2 \, dx$$

$$= \int_0^2 4x - 2x^2 \, dx$$

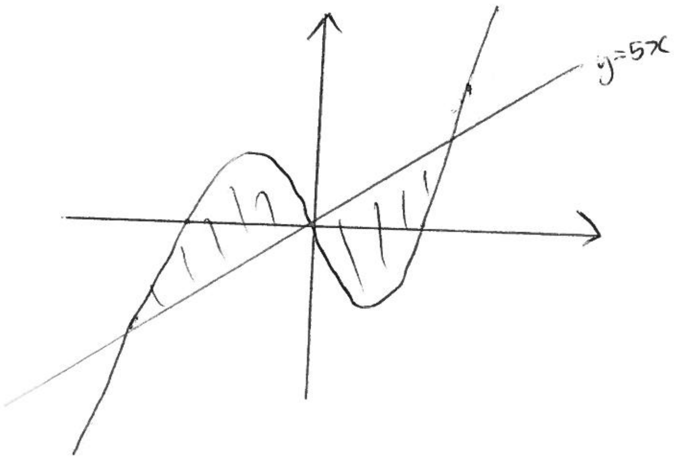
$$= \left[\frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2$$

$$= \left(8 - \frac{16}{3} \right) - 0$$

$$= \frac{8}{3} \text{ sq units}$$

eg: Find the area between the curves $y = x^3 - 4x$ and $y = 5x$

Draw:



$$y = x^3 - 4x$$
$$= x(x^2 - 4)$$
$$= x(x+2)(x-2)$$

Find pt of int:

$$x^3 - 4x = 5x$$
$$x^3 - 9x = 0$$
$$x(x^2 - 9) = 0$$
$$x = 0, \pm 3$$

$$\text{Area} = 2 \times \int_0^3 5x - (x^3 - 4x)\, dx$$

$$= 2 \times \int_0^3 9x - x^3\, dx$$

$$= 2 \times \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3$$

$$= 2 \times \left[\left(\frac{81}{2} - \frac{81}{4} \right) - 0 \right]$$

$$= \frac{81}{2} \text{ sq units}$$