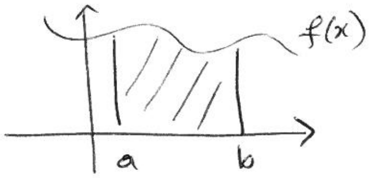
Areas + Integration

So Far:



Integration → finding area under curve

$$\int_a^b f(x)\,dx$$

Opposite of Differentiation

We concentrated on finding anti-derivatives

eg i) $\int \frac{1}{x^3} - \frac{4}{x} + e^{5x}\,dx = \int x^{-3} - \frac{4}{x} + e^{5x}\,dx$

$= \frac{x^{-2}}{-2} - 4\log|x| + \frac{1}{5}e^{5x} + c$

2) $\int \frac{1}{7x-5}\,dx = \frac{1}{7}\log|7x-5| + c$

3) $\int \sin 4x\,dx = -\frac{1}{4}\cos 4x + c$

4) $\int 4\cos(3-2x)\,dx = \frac{4}{-2}\sin(3-2x) + c$

$= -2\sin(3-2x) + c$

6) $\displaystyle\int_0^4 \sqrt{3x+4}\,dx = \int_0^4 (3x+4)^{1/2}\,dx$

$$= \frac{1}{3}\,\frac{(3x+4)^{3/2}}{3/2}\Bigg|_0^4$$

$$= \frac{2}{9}\left[(3x+4)^{3/2}\right]_0^4$$

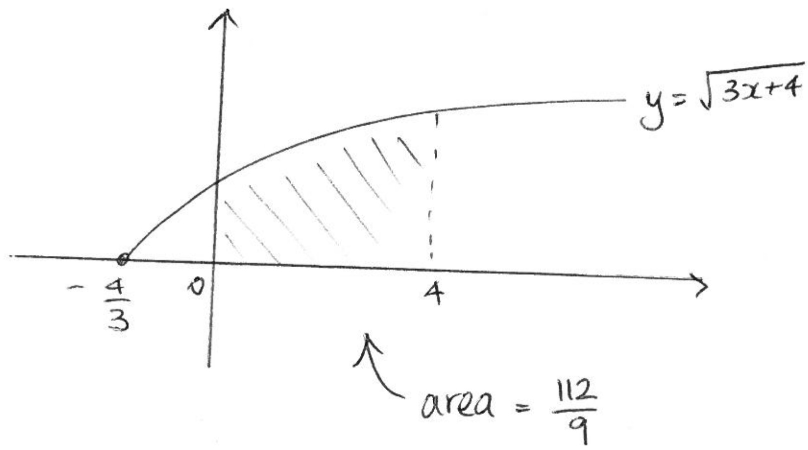
$$= \frac{2}{9}\left(16^{3/2} - 4^{3/2}\right)$$

$$= \frac{2}{9}\left(4^3 - 2^3\right)$$

$$= \frac{2}{9}(64-8)$$

$$= \frac{112}{9}$$

We can picture this



What about

$$\int_0^2 (x-2)^3 \, dx = \left[\frac{(x-2)^4}{4} \right]_0^2$$

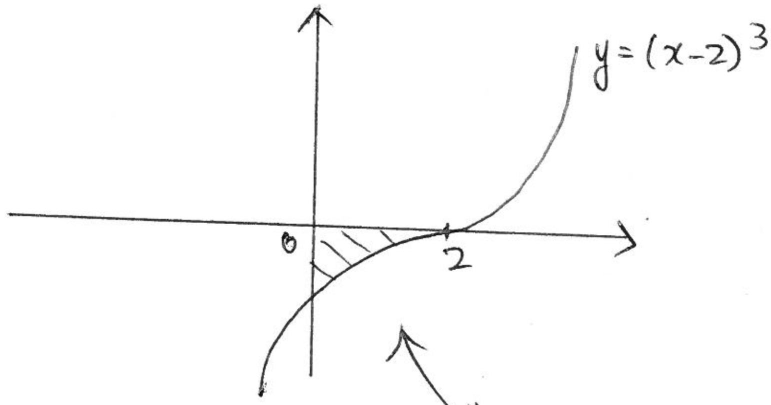
$$= \frac{1}{4} \left[(x-2)^4 \right]_0^2$$

$$= \frac{1}{4} \left(0 - (-2)^4 \right)$$

$$= \frac{1}{4} (-16)$$

$$= -4$$

negative area??

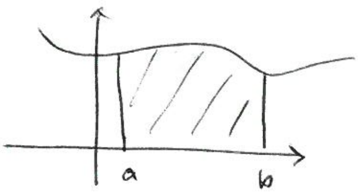


Picture whats happening

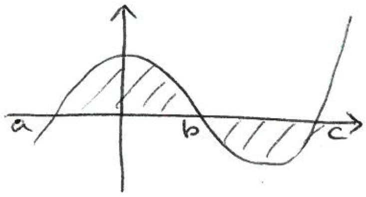
"Area" below x axis is negative

Areas under Curves

Infact : Integral = DIRECTED area under a curve



$\int_a^b f(x)\,dx$ ← Here $f(x) > 0$

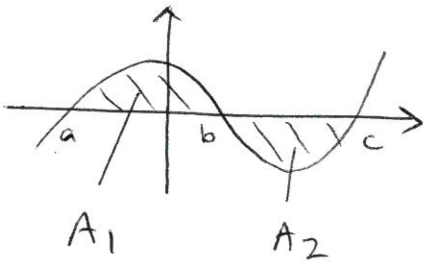


areas above x-axis → Positive

areas below x-axis → Negative

∴ Here Integral $\int_a^c f(x)\,dx$ adds areas above and subtracts areas below

∴ We must be careful if we actually want the area under a curve.



Integral : $\int_a^c f(x)\,dx = A_1 - A_2$

∴ Area ← we want $A_1 + A_2$

↳ need to split this

$$= \int_a^b f(x)\,dx + \left| \int_b^c f(x)\,dx \right|$$

eg a) Find the integral $\int_0^5 (2x-4)\,dx$

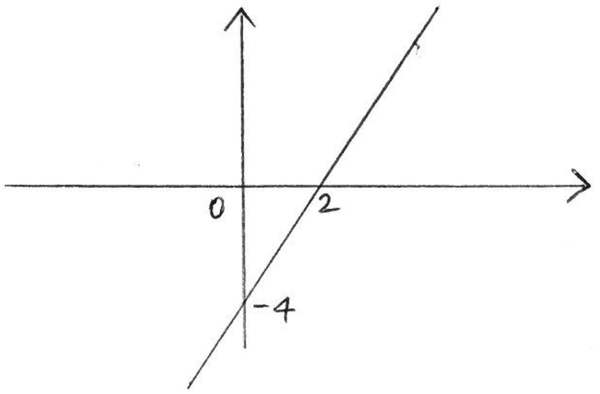
$$\int_0^5 2x-4\,dx = \left[\frac{\cancel{2}x^2}{\cancel{2}} - 4x\right]_0^5$$

$$= (25-20) - 0$$

$$= 5$$

b) Find the area between $y=2x-4$ + the x-axis between $x=0$ and $x=5$

ie!



Find x intercept $\rightarrow$ Let $y=0$: $2x-4=0$

$x=2$

$$\therefore \text{Area} = \left|\int_0^2 2x-4\,dx\right| + \int_2^5 2x-4\,dx$$

$$= \left|\left[x^2-4x\right]_0^2\right| + \left[x^2-4x\right]_2^5$$

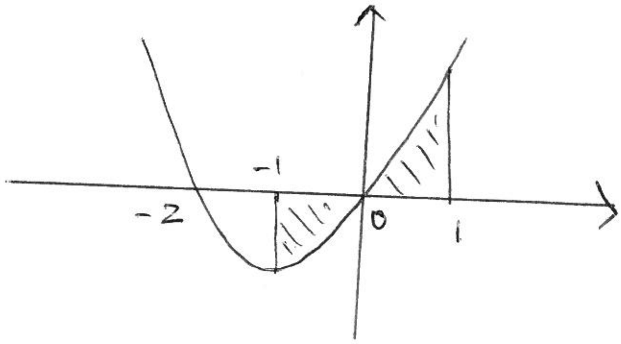
$$= \left|\left((4-8)-0\right)\right| + \left((25-20)-(4-8)\right)$$

$$= |-4| + (5-(-4))$$

$$= 4 + 9 = 13 \text{ sq units}$$

Eg) Find the area between the curve $y = x^2 + 2x$, the x-axis and the lines $x = -1$ and $x = 1$.

First picture this: $y = x^2 + 2x = x(x+2)$



$$\text{Area} = \left| \int_{-1}^{0} x^2 + 2x \, dx \right| + \int_{0}^{1} x^2 + 2x \, dx$$

$$= \left| \left[\frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} \right]_{-1}^{0} \right| + \left[\frac{x^3}{3} + x^2 \right]_{0}^{1}$$

$$= \left| 0 - \left(-\frac{1}{3} + 1 \right) \right| + \left(\frac{1}{3} + 1 \right) - 0$$

$$= \left| -\frac{2}{3} \right| + \frac{4}{3}$$

$$= \frac{2}{3} + \frac{4}{3}$$

$$= \frac{6}{3}$$

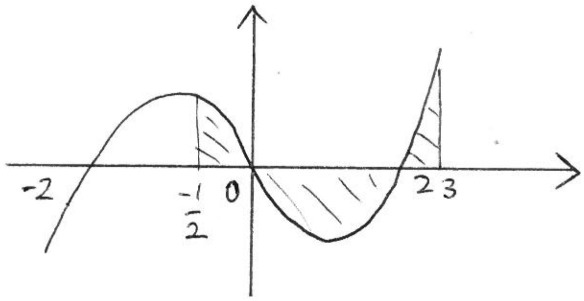
$$= 2$$

Eg) Find the area between the curve $y = x^3 - 4x$ and the x-axis between $x = -1/2$ and $x = 3$.

Draw: $y = x^3 - 4x$

$= x(x^2 - 4)$

$= x(x+2)(x-2)$



$$\text{Area} = \int_{-1/2}^{0} x^3 - 4x \, dx + \left| \int_0^2 x^3 - 4x \, dx \right| + \int_2^3 x^3 - 4x \, dx$$

$$= \left[\frac{x^4}{4} - \frac{4x^2}{2} \right]_{-1/2}^{0} + \left| \left[\frac{x^4}{4} - 2x^2 \right]_0^2 \right| + \left[\frac{x^4}{4} - 2x^2 \right]_2^3$$

$$= \left(0 - \left(\tfrac{1}{4}\left(\tfrac{1}{16}\right) - 2\left(\tfrac{1}{4}\right) \right) \right) + \left| \left(\frac{16}{4} - 8 \right) - 0 \right| + \left(\left(\frac{81}{4} - 18 \right) - \left(\frac{16}{4} - 8 \right) \right)$$

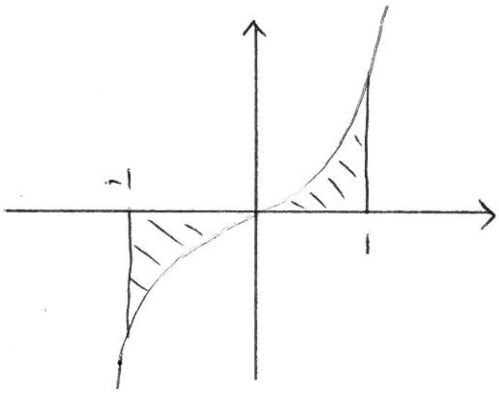
$$= \frac{31}{64} + |-4| + \frac{25}{4}$$

$$= \frac{31}{64} + 4 + \frac{25}{4}$$

$$= \frac{687}{64} \text{ sq units}$$

eg) Find the area between $y = x^3$, the x-axis and the lines $x = 1$ and $x = -1$

Draw



Area $= \left| \int_{-1}^{0} x^3 \, dx \right| + \int_{0}^{1} x^3 \, dx$

OR $2 \int_0^1 x^3 \, dx$ ← Look out for symmetry! (This is an odd function)

$= 2 \left[\frac{x^4}{4} \right]_0^1$

$= 2 \left(\frac{1}{4} \right)$

$= \frac{1}{2}$ sq units

(Note: Integral $\int_{-1}^{1} x^3 \, dx = 0$ since areas cancel by symmetry)

Look out for even + odd functions