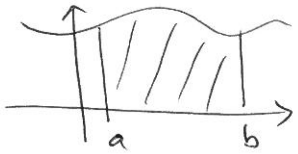
More Integration

Refresher

We started finding the area under a curve



$\leftarrow \int_a^b f(x)\,dx$

$\uparrow$

Integration

↳ opposite of differentiation

Fundamental Theorem: If $F' = f$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

$\nearrow$

F = anti derivative of f

We saw $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

$\int k \, dx = kx + C$

eg 1) $\int x^3 - 4x^2 + 6x - 11 \, dx = \dfrac{x^4}{4} - \dfrac{4x^3}{3} + \dfrac{6x^2}{2} - 11x + C$

2) $\int 2 - \dfrac{3}{x^4} - \sqrt{x} \, dx = \int 2 - 3x^{-4} - x^{1/2} \, dx$

$= 2x - 3\dfrac{x^{-3}}{-3} - \dfrac{2}{3}x^{3/2} + C$

$= 2x + \dfrac{3}{x^3} - \dfrac{2}{3}x^{3/2} + C$

$\int f(x)\,dx$ ← indefinite integral (ie: finding antiderivatives)

$\int_a^b f(x)\,dx$ ← definite integral ← Looking at area between curve and x axis on $[a,b]$

↑ We use antiderivatives to help us get this.

Definite Integrals

Fund Th: If $F' = f$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

↑ a, b are called limits

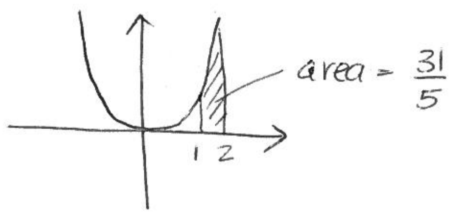
↑ after finding antideriv we sub limits in (top − bottom)

eg: $\int_1^2 x^4\,dx = \left[\frac{x^5}{5}\right]_1^2$

$= \frac{2^5}{5} - \frac{1^5}{5}$

$= \frac{32}{5} - \frac{1}{5}$

$= \frac{31}{5}$



note: we don't worry about the const for definite integrals. (it will cancel out anyway)

ie: $\int_1^2 x^4\,dx = \left[\frac{x^5}{5} + c\right]_1^2 = \left(\frac{2^5}{5} + c\right) - \left(\frac{1}{5} + c\right) = \frac{2^5}{5} - \frac{1}{5}$

1 a) $\int_0^2 4 - x^2 \, dx = \left[4x - \frac{x^3}{3}\right]_0^2$

$= \left(4(2) - \frac{2^3}{3}\right) - 0$

$= \frac{16}{3}$

b) $\int_1^{25} \frac{1}{\sqrt{x}} \, dx = \int_1^{25} x^{-1/2}$

$= \left.\frac{x^{1/2}}{1/2}\right|_1^{25}$ ← another notation

$= \left.2x^{1/2}\right|_1^{25}$

$= \left.2\sqrt{x}\right|_1^{25}$

$= 2\sqrt{25} - 2\sqrt{1}$

$= 2(5) - 2$

$= 8$

c) $\int_{-1}^{7} 3x - \frac{1}{x^2} \, dx = \int_{-1}^{7} 3x - x^{-2} \, dx$

$= \left[\frac{3x^2}{2} - \frac{x^{-1}}{-1}\right]_{-1}^{7}$

$= \left[\frac{3x^2}{2} + \frac{1}{x}\right]_{-1}^{7}$

$= \left(\frac{3(49)}{2} + \frac{1}{7}\right) - \left(\frac{3}{2} - 1\right) = \frac{512}{7}$

d) $\int_1^2 x - x^2 \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_1^2$

$= \left(\frac{4}{2} - \frac{8}{3} \right) - \left(\frac{1}{2} - \frac{1}{3} \right)$

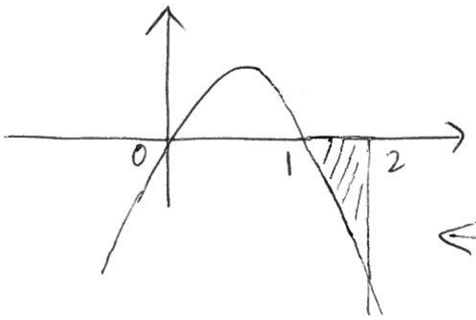
$= -\frac{2}{3} - \frac{1}{6}$

$= -\frac{5}{6}$

notice neg answer

But we're talking about areas...

Here

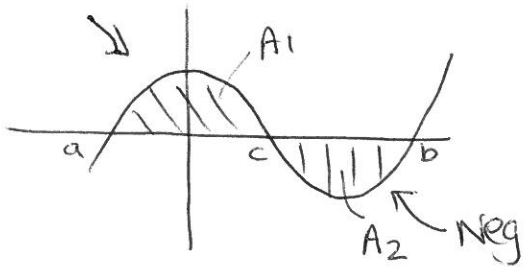


$y = x - x^2$
$= x(1-x)$

"area" $= -\frac{5}{6}$

Infact $\int_a^b f(x) \, dx$ gives the directed area

areas above x axis positive
" below " " neg.



Evaluating $\int_a^b f(x) \, dx$ will add A_1 and subtract A_2

(impt later when finding areas)

Later we will look at areas, but for now we will concentrate on evaluating definite integrals.

Properties of Definite Integrals

① $$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$F(b) - F(a)$

$-\left(F(a) - F(b)\right)$

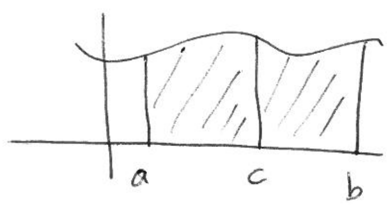
$= F(b) - F(a)$

← can easily see this.

② $$\boxed{\int_a^a f(x)\,dx = 0}$$

③ $$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

for $a < b$ and $c =$ any n^o between $a + b$

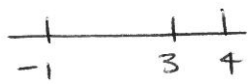


④ Also we have seen.

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Eg 2) If $\int_{-1}^{4} f(x)\,dx = 3$ and $\int_{3}^{4} f(x)\,dx = 7$, Find $\int_{-1}^{3} f(x)\,dx$.

So we have 

and $\int_{-1}^{4} = \int_{-1}^{3} + \int_{3}^{4}$

$\therefore \int_{-1}^{3} = \int_{-1}^{4} - \int_{3}^{4}$

$$\therefore \int_{-1}^{3} f(x)\,dx = \int_{-1}^{4} f(x)\,dx - \int_{3}^{4} f(x)\,dx$$

$$= 3 - 7$$

$$= -4$$

An Integration Technique

Reverse Chain Rule

We saw integration = opposite of differentiation

ie: $\frac{d}{dx} x^3 = 3x^2 \rightarrow \int 3x^2\,dx = x^3$

ie: $\int x^2\,dx = \frac{1}{3}x^3 + c$

$\frac{d}{dx}(2x+1)^3 = 2\cdot 3(2x+1)^2 \longrightarrow \int 2\cdot 3(2x+1)^2\,dx = (2x+1)^3 + c$

ie: $\int (2x+1)^2\,dx = \frac{1}{2}\frac{(2x+1)^3}{3} + c.$

$\frac{d}{dx}(5x-4)^3 = 5\cdot 3(5x-4)^2 \longrightarrow \int (5x-4)^2\,dx = \frac{1}{5}\frac{(5x-4)^3}{3} + c$

$\frac{d}{dx}(10x+1)^3 = 10\cdot 3(\quad)^2 \longrightarrow \int (10x+1)^2\,dx = \frac{1}{10}\frac{(10x+1)^3}{3} + c$

Notice the pattern $\quad (ax+b)^2 \quad \frac{1}{a}\quad \frac{(\quad)^3}{3}$

$\frac{d}{dx}(10x+1)^7 = 10\cdot 7(\quad)^6 \longrightarrow \int (10x+1)^6\,dx = \frac{1}{10}\frac{(\quad)^7}{7}$

$= \frac{1}{10}\frac{(10x+1)^7}{7} + c$

$\frac{1}{a}$ $\quad$ integ of "main function"