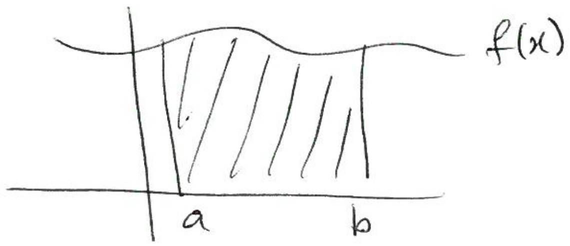
Yesterday we saw



$$\text{Area} = A = \int_a^b f(x)\,dx$$

area between $f(x)$ + x axis
between a + b.

Then we proved

$$A' = f(x)$$

ie: Differentiation "undoes" integration

Integration = opposite of differentiation

- This can all be neatly wrapped up by the following theorem:

The Fundamental Theorem of Calculus

$$\boxed{\text{If } \quad F'(x) = f(x) \quad \text{, then} \quad \int_a^b f(x) = F(b) - F(a)}$$

Proof: . Let $F'(x) = f(x)$

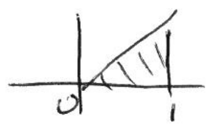
. Since functions with same derivatives differ by a constant then $F(x) = A(x) + c$.

. So $F(a) = A(a) + c$
$F(b) = A(b) + c$

$$\therefore \int_a^b f(x)\, dx = A(b) - A(a)$$
$$= A(b) + c - (A(a) + c)$$
$$= F(b) - F(a).$$

$\blacksquare$

eg: Find $\int_0^1 x \, dx$ $\leftarrow$ area under curve $y = x$



Using Fund Th: $\int_0^1 x \, dx = \left[\frac{x^2}{2} \right]_0^1$ $\leftarrow$ find function whose deriv is $f(x) = x$

$$= \frac{1}{2} - \frac{0}{2}$$
$$= \frac{1}{2}$$

Integration

- The opposite of differentiation

eg:
$$\begin{tikzcd} x^2 \arrow[r, bend left, "\text{diff}"] & 2x \arrow[l, bend left, "\text{integ}"] \end{tikzcd}$$

- Suppose $f(x)$ is our given function.
- If $F'(x) = f(x) \Rightarrow F(x)$ is the $\left\{\begin{array}{l}\text{anti derivative of } f(x)\\ \text{primitive}\\ \text{integral}\end{array}\right.$

- eg: $f(x) = 2x$

  We know $\dfrac{d}{dx}(x^2) = 2x$

  $\therefore$ The anti derivative of $2x$ is x^2

  But notice $2x$ has other antiderivatives.

  $x^2 + 1$

  $x^2 - 20$

  $\vdots$

$\left\{\begin{array}{l}\text{Integral}\\ \text{Antiderivative}\end{array}\right.$ is not unique, but they only differ by a constant.

$\therefore$ We write $\displaystyle\int f(x)\,dx = \underset{\text{antideriv}}{F(x)} + \underset{\text{since they differ by a constant}}{C}$

- Integration = process of finding the antiderivative

$$\int f(x)\, dx \quad \leftarrow \text{indefinite integral}$$

$$\int_a^b f(x)\, dx \quad \leftarrow \text{definite integral}$$

- we will first concentrate of indefinite integrals and integrating.
- we can use our knowledge of differentiation to establish some basic rules.

- Notice patterns.

$$x^2 \underset{\text{integ}}{\overset{\text{diff}}{\rightleftarrows}} 2x$$

integ of $2x$ is x^2

integ of x is $\frac{x^2}{2}$ (since $\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{\cancel{2}x}{\cancel{2}} = x$)

integ of x^3 is $\frac{x^4}{4}$

integ of x^5 is $\frac{x^6}{6}$

ie: $\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + c}$ $(n \neq -1)$

ie: $\int x \, dx = \frac{x^2}{2} + c$

$\int x^3 \, dx = \frac{x^4}{4} + c$

$\int x^5 \, dx = \frac{x^6}{6} + c$

A few things to note:

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ ← only use this for powers of x

  n must be a number

  Don't do $\cancel{\int 2^x = \frac{2^{x+1}}{x+1}}$

- $\int f(x) \, dx$

  ↑ this tells us what we're integrating ie: f is a function of x + apply integ rules to x.

  $\int u^3 \, dx$ ← doesn't make sense

  But $\int u^3 \, du = \frac{u^4}{4} + c$

Egs

1) $\int x^7 \, dx = \frac{x^8}{8} + c$

2) $\int x^{-3} \, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c$

3) $\int \frac{1}{x^5} \, dx = \int x^{-5} \, dx$

$= \frac{x^{-4}}{-4} + c$

4) $\int \sqrt{x} \, dx = \int x^{1/2} \, dx$

$= \frac{x^{3/2}}{3/2} + c$

$= \frac{2}{3} x^{3/2} + c$

5) $\int x^{1/5} \, dx = \frac{x^{6/5}}{6/5} + c$

$= \frac{5}{6} x^{6/5} + c$

6) $\int x^{-2/3} \, dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + c$

$= \frac{x^{1/3}}{1/3} + c$

$= 3x^{1/3} + c$

Try Q2

More egs:

1) $\int 2x \, dx = 2\int x \, dx$ ← can take constants out front

$= 2\left[\frac{x^2}{2}\right] + c$

$= x^2 + c$

2) $\int 7x^4 \, dx = 7\int x^4 \, dx$

$= 7\left[\frac{x^5}{5}\right] + c$

$= \frac{7x^5}{5} + c$

3) $\int 3x^{-4} \, dx = 3\left[\frac{x^{-3}}{-3}\right] + c$

$= -x^{-3} + c$

4) $\int 4x^{1/3} \, dx = 4\int x^{1/3} \cdot dx$

$= 4\left[\frac{x^{2/3}}{2/3}\right] + c$

$= 4 \cdot \frac{3}{2} \cdot x^{2/3} + c$

$= 6x^{2/3} + c$

5) $\int \sqrt{8x} \, dx = \sqrt{8}\int x^{1/2} \, dx$

$= \sqrt{8}\,\frac{x^{3/2}}{3/2} + c = \frac{2\sqrt{8}}{3}x^{3/2} + c$