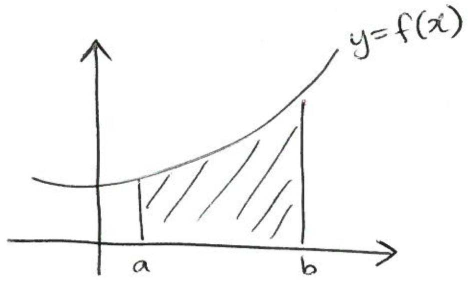
Integration

- the second process of calculus
- finds areas between the curve and x-axis.

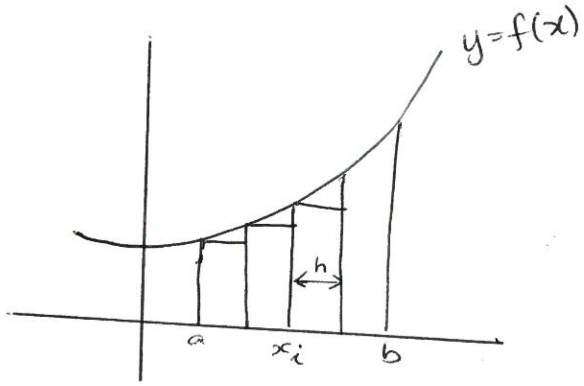
eg:



← we want this area.

How do we do it?

- **Big Picture**:



Start by estimating using rectangles

→ divide each rectangle in width $h = \Delta x$

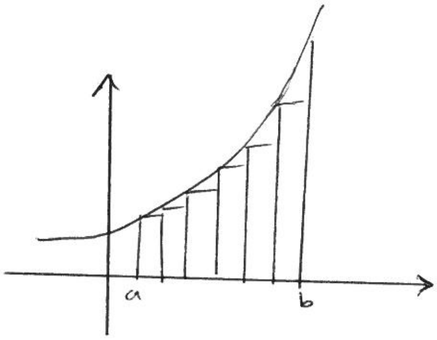
→ area of each rectangle = height × width

$$= f(x_i) \times \Delta x$$

→ Area = sum of areas of rectangles

$$= \sum f(x_i)\,\Delta x$$

→ we can make this estimate better by making width Δx smaller (ie: more rectangles)



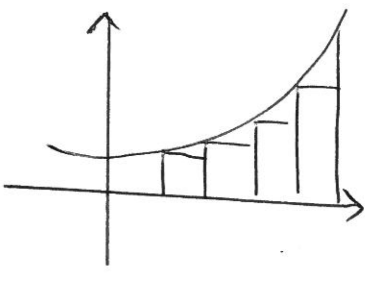
ie: $$\lim_{\Delta x \to 0} \sum f(x_i)\, \Delta x$$

ie: $$\int_a^b f(x)\, dx$$

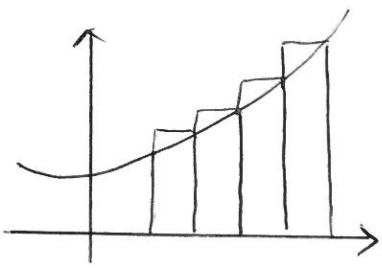
- we use this integral sign to represent this limiting sum
- the integral symbol {represents / is defined as} this perfect area under a curve

- The Details

- There are different ways of dividing up the rectangles



Lower Rectangles L_4



Upper Rectangles U_4

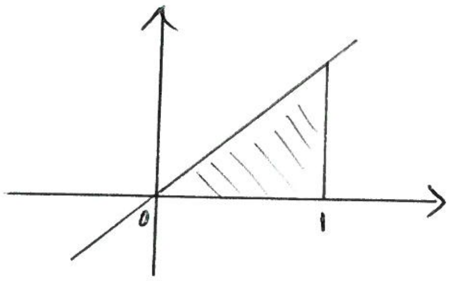
We can see $L_4 \leq \text{True Area} \leq U_4$

$$L_4 \rightarrow \underset{A}{\overset{\text{True Area}}{\|}} \leftarrow U_4 \quad \text{as } \Delta x \to 0$$

As $\Delta x \to 0$, L_4 + U_4 will both approach A.

eg: $f(x) = x$

ie: $y = x$



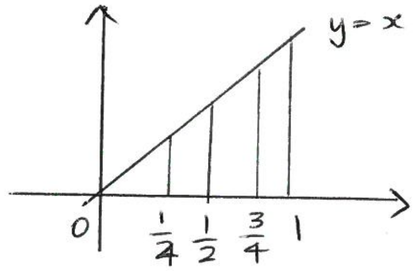
Look at area under curve $y=x$ between 0 and 1.

Calculating L_4 + U_4

4 subdivisions

width $= \frac{1-0}{4}$

height = function value



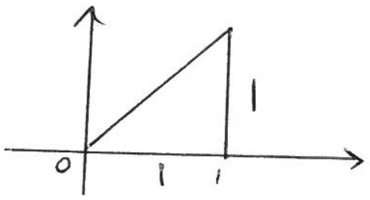
$$L_4 = \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$$
$$= \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right)$$
$$= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right)$$
$$= \frac{1}{4} \cdot \frac{3}{2}$$
$$= \frac{3}{8}.$$

$$U_4 = \frac{1}{4}\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1)\right)$$
$$= \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1\right)$$
$$= \frac{1}{4}\left(\frac{5}{2}\right) = \frac{5}{8}$$

We can see $\frac{3}{8} \leq \underset{\text{area}}{\text{true}} \leq \frac{5}{8}$

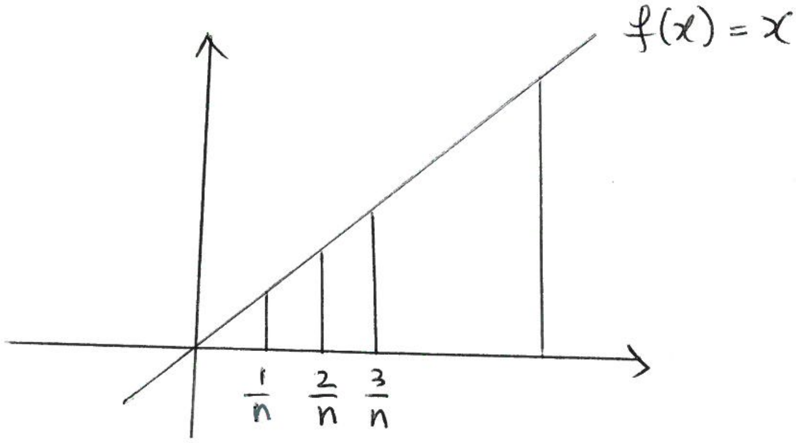
Infact in this case we can calculate true area using our knowledge of triangles.

ie:



$$\text{Area} = \frac{1}{2} bh$$
$$= \frac{1}{2} \times 1 \times 1$$
$$= \frac{1}{2}$$

But now lets prove that lower Rect (+upper) do approach the true area.



Take n rectangles
- they have width $\frac{1}{n}$
- their height is the function value $f\left(\frac{k}{n}\right)$

$$\text{Area} = \lim_{n\to\infty} \left(\frac{1}{n} f(0) + \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n-1}{n}\right) \right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) \right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n} \right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \left(1 + 2 + \cdots + (n-1) \right)$$

AP with $a=1$, $d=1$, n terms

$$= \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{n}{2} \left(2 + (n-1)(1)\right) \right)$$

formula for sum of AP

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n}{2} (1+n)$$

$$= \lim_{n\to\infty} \frac{1+n}{2n}$$

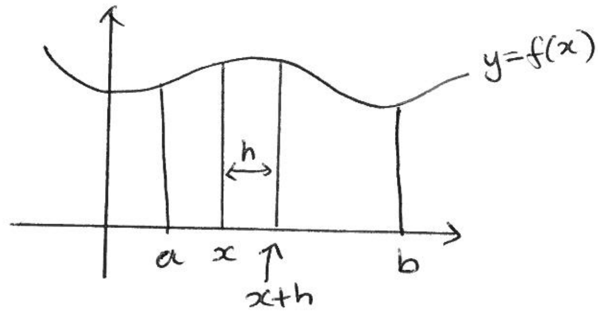
$$= \lim_{n\to\infty} \frac{1}{2n} + \frac{\cancel{n}}{2\cancel{n}}$$

$$= 0 + \frac{1}{2}$$

$$= \frac{1}{2}$$

First principles approach

- $f(x)$ on interval $[a,b]$
- n rectangles.
- width of rectangle $= h$
- typical strip - x to $x+h$

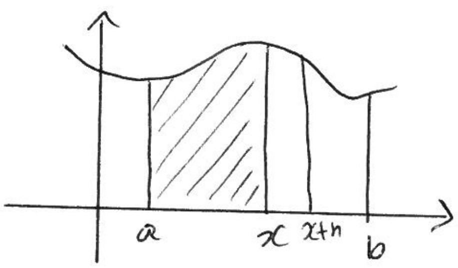


Aim: To find the true area under the curve

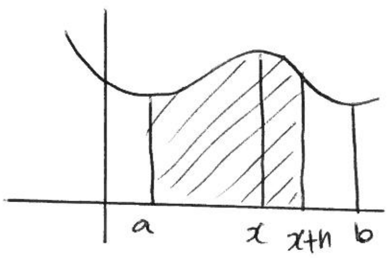
ie: Want to find $\int_a^b f(x)\,dx$.

Let $A(x)$ = true area under curve on $[a,x]$

ie:



So $A(x+h)$



$\therefore$ Area of strip $= A(x+h) - A(x)$

Now we'll find lower + upper bounds

Let m = min height of strip
M = max " "

then Lower Rectange area $= mh$
Upper " " $= Mh$

and we know Lower $\leq$ true area $\leq$ upper

ie: $$mh \leq A(x+h) - A(x) \leq Mh$$

ie: $$m \leq \frac{A(x+h) - A(x)}{h} \leq M$$

as $h \to 0$, m and $M \to f(x)$

so $$\begin{array}{ccccc} m & \leq & \dfrac{A(x+h) - A(x)}{h} & \leq & M \\ \downarrow & & & & \downarrow \\ f(x) & & & & f(x) \end{array}$$

$$\therefore \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

But this is the definition of the derivative!

ie: $$A'(x) = f(x)$$

derivative of area function

so
$$\begin{tikzcd} f(x) \arrow[r, bend right, "\text{integ}"'] & A(x) \arrow[l, bend right, "\text{diff}"'] \end{tikzcd}$$

integ $\leftarrow$ since we defined integ as finding the area.

$\therefore$ Process of finding area = opposite of differentiation

ie: Integration = opposite of differentiation