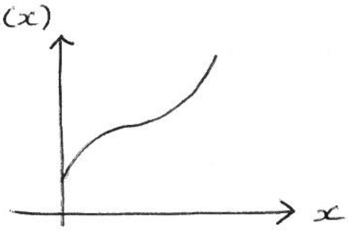
Application to Economics

- Calculus can be used in certain applications of economics, in particular in management decisions involving cost and revenue.

Cost function

$C(x)$ = total cost of producing x units of certain commodity



$C'(x)$ = rate of change of cost with respect to production level x.

= marginal cost

This is different to the average cost $= \frac{C(x)}{x}$

Eg: A company estimates that the cost (in dollars) of producing x items is $C(x) = 2600 + 2x + 0.001x^2$

The marginal cost is $C'(x) = 2 + 0.002x$

So when producing 1000 units the marginal cost is

$C'(1000) = 2 + 0.002(1000) = 4$ ie: \$4.

ie: when producing 1000 units, it will cost \$4 to produce the next unit of output.

Average cost $= \frac{C(x)}{x} = \frac{2600}{x} + 2 + 0.001x$

When producing 1000 units, Avg cost $= \frac{2600}{1000} + 2 + 0.001(1000)$

$= 5.6$

ie: \$5.60

ie: when prod 1000 units, it has cost \$5.60 so far.

The most efficient production level occurs when we minimise the average cost.

$A =$ Avg cost $= 2600x^{-1} + 2 + 0.001x$

$A' = -2600x^{-2} + 0.001$

$A'' = 5200x^{-3}$

Find min stat pt: $A' = 0$

$-\frac{2600}{x^2} + 0.001 = 0$

$\frac{2600}{x^2} = 0.001$

$x^2 = \frac{2600}{0.001} = 2600\,000$

$x = \sqrt{2600000} = 1612$

check this is min: $A'' = \frac{5200}{(1612)^3} > 0$ ∪ min ✓

∴ Avg cost is min when production level is 1612

\+ avg cost is $\frac{C(1612)}{1612} = 2600 + 2 + 0.001(1612)$

$= 5.22$

eg: A firm estimates that the cost of producing x units is $C(x) = 3400 + 4x + 0.002x^2$.

a) What is the marginal cost when $x = 500$?

b) What is the minimum avg cost + at what production level is this achieved?

a) $C(x) = 3400 + 4x + 0.002x^2$

marginal cost $= C'(x) = 4 + 0.004x$

when $x = 500$, $C'(x) = 4 + 0.004(500) = \$6$

b) Avg cost $= \dfrac{C(x)}{x} = \dfrac{3400}{x} + 4 + 0.002x$

Let $A = 3400x^{-1} + 4 + 0.002x$

$A' = -3400x^{-2} + 0.002$

$A'' = 6800x^{-3}$

Find min: $A' = 0$

$$-\frac{3400}{x^2} + 0.002 = 0$$

$$x^2 = \frac{3400}{0.002} = 1700000$$

$$x = \sqrt{1700000} = 1304$$

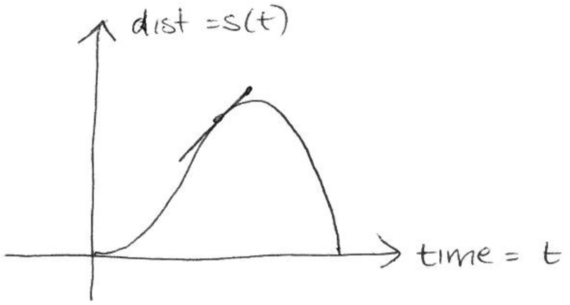
$\therefore$ Min Avg cost occurs when production is 1304 and is $\dfrac{C(1304)}{1304} = \dfrac{3400}{1304} + 4 + 0.002(1304)$

$= \$9.22$

Application - Motion in a straight line

Remember policeman story:

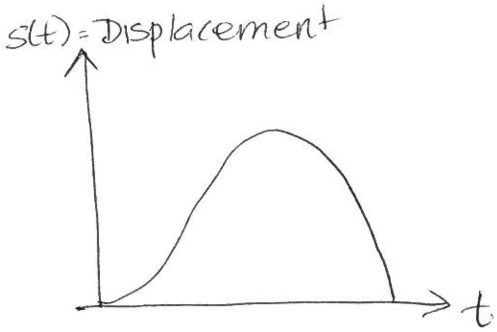
We said:



$$\text{Speed} = \frac{\text{change in dist}}{\text{change in time}}.$$

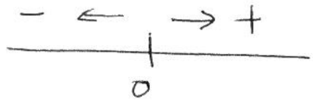
$$\text{Instantaneous speed} = \frac{ds}{dt}$$

Now



$$\text{Velocity} = \frac{ds}{dt}$$

$s(t)$ = Displacement ← like distance but has a direction



It tells us the position

$s = +4$ ← 4 units to right of start
$S = -4$ ← " " " left " "

$\frac{ds}{dt}$ = velocity ← like speed but has a direction
vel pos → travelling in pos direction.
vel neg → " " neg "

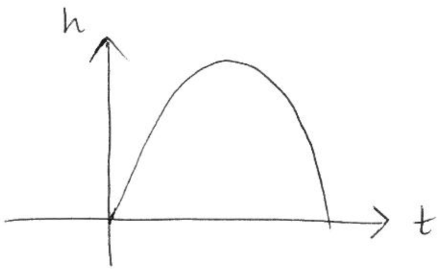
Also $\frac{d^2s}{dt^2}$ = change in vel = acceleration
↑ like a force.
pos/neg - think of it as the direction you are being "pushed" in

(vel + accel have same sign → faster
" " " diff " → slower)

7. A ball is thrown vertically up in the air and its position relative to the ground after t seconds is given by the functions $h(t) = 100t - 5t^2$.

   (a) Find the velocity and acceleration functions.
   (b) What is the initial velocity?
   (c) For how long will the ball rise?
   (d) How high will the ball rise?
   (e) Describe the motion of the ball at time $t = 5$.
   (f) Find the time for the ball to reach the ground.

Firstly $h(t) = 100t - 5t^2$ ← we know what this looks like



$h(t) = 5t(20 - t)$

a) vel $= h'(t) = 100 - 10t$

accel $= h''(t) = -10$

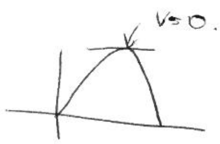
b) Initial velocity → want v when $t = 0$.

ie: vel $= 100 - 10(0)$

$= 100$ m/s

c) want time where vel is positive.

ie: time for it to take to get to $v = 0$



∴ Let vel $= 0$

$100 - 10t = 0$

$t = 10$ secs

d) max height is reached when $v=0$

ie: when $t=10$

$\therefore$ want height when $t=10$

$$h(10) = 100(10) - 5(10)^2$$
$$= 1000 - 500$$
$$= 500 \text{ m}$$

e) At $t=5$:

disp $= h(5) = 100(5) - 5(5)^2 = 375$

vel $= h'(5) = 100 - 10(5) = 50$

accel $= h''(5) = -10$

$\therefore$ The ball is 375 m above ground level and is travelling at a velocity of 50 m/s. Since the acceleration is neg it is slowing down.

f) Ball reaches ground when $h(t)=0$

ie: want t when $h(t)=0$.

$$100t - 5t^2 = 0$$
$$5t(20-t) = 0$$
$$\therefore t = 0, 20$$

start.

$\therefore$ It reaches the ground after 20 secs.

8. Consider a particle moving in a straight line with displacement given by $s(t) = \frac{t^3}{3} - 3t^2 + 5t + 4$, where s is measured in metres and t in seconds.

(a) What is the initial velocity of the particle?

(b) What is the velocity and acceleration at $t = 4$?

(c) Describe the motion of the particle at $t = 4$.

(d) When does the particle come to rest?

$$s(t) = \frac{t^3}{3} - 3t^2 + 5t + 4$$

vel $= s'(t) = t^2 - 6t + 5$

accel $= s''(t) = 2t - 6$

a) initial vel $\rightarrow$ $s'(0) = 5$ m/s

b) At $t = 4$: vel $= s'(4) = 4^2 - 6(4) + 5 = -3$ m/s

accel $= s''(4) = 8 - 6 = 2\ \text{ms}^{-2}$

c) Particle is travelling at 3 m/s in the negative direction. Since accel is positive it is slowing down.

d) At rest when vel $= 0$

ie: $t^2 - 6t + 5 = 0$

$(t-5)(t-1) = 0$

$t = 1, 5$

$\therefore$ Particle comes to rest at $t = 1$, $t = 5$.