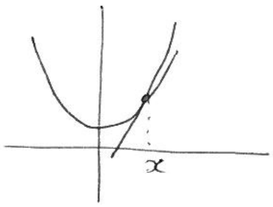
Applications of Differentiation

Refresher:

- Derivative = slope of tangent at each point x on curve



- If $f'(x) > 0 \Rightarrow f(x)$ increasing
  $f'(x) < 0 \Rightarrow f(x)$ decreasing
  $f'(x) = 0 \Rightarrow f(x)$ stationary

- Stationary point = point where slope is 0
  = " " derivative is 0
  $\Rightarrow$ Solve $f'(x)=0$ to find statpts.

  3 types:

max | min | horiz pts of inflexion

- Classifying stat pts:
  - 1st derivative test $\rightarrow$ test slope on either side of stat pt
  - 2nd derivative test $\rightarrow$ tells us about concavity

  If $f''(x) > 0 \Rightarrow$  concave up minimum

  $f''(x) < 0 \Rightarrow$  concave down maximum

  $f''(x) = 0 \Rightarrow$ No info about stat pt

- 2nd Derivative – tells us whether curve is concave up or concave down

: Points of inflexion = point where concavity changes.

$\Rightarrow$ Find by solving $f''(x) = 0$ + check concavity changes

Sketching curves:

1. Find stat pts
2. Classify them
3. Points of inflexion
4. Intercepts
5. Behaviour.

eg 1a) Sketch the graph of $y = x^4 - 2x^2 + 7$

$y' = 4x^3 - 4x$

$y'' = 12x^2 - 4$

stat pts: $4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$x = 0, 1, -1$

when $x = 0$: $y = 7 \rightarrow (0,7)$

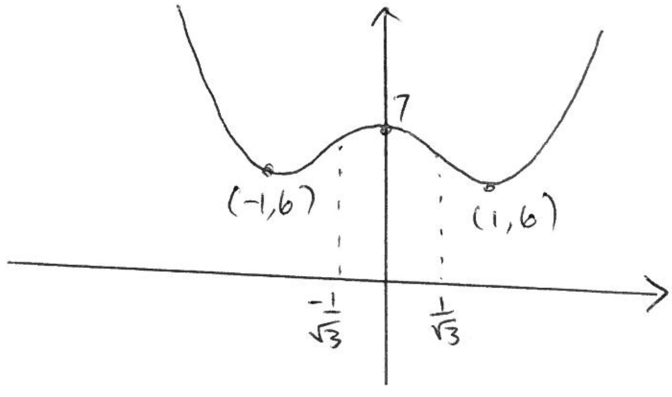
$x = 1$: $y = 1 - 2 + 7 = 6 \rightarrow (1,6)$

$x = -1$: $y = 6 \rightarrow (-1,6)$

nature: $(0,7)$: $y'' = -4 < 0$ $\cap$ max

$(1,6)$: $y'' = 12 - 4 > 0$ $\cup$ min

$(-1,6)$: $y'' = 12 - 4 > 0$ $\cup$ min



pts of inflex:

$12x^2 - 4 = 0$

$12x^2 = 4$

$x^2 = \frac{4}{12} = \frac{1}{3}$

$x = \pm \frac{1}{\sqrt{3}}$

eg 16) Sketch the graph of $y = -3x^5 - 10x^3 - 18x + 31$

$y' = -15x^4 - 30x^2 - 18$

$y'' = -60x^3 - 60x$

st pts: $y' = 0$: $-15x^4 - 30x^2 - 18 = 0$

Let $m = x^2$: $-15m^2 - 30m - 18 = 0$

ie: $-3(5m^2 + 10m + 6) = 0$

$5m^2 + 10m + 6 = 0$

Notice $\Delta = 100 - 4(5)(6) < 0$

$\therefore$ No solution

$\therefore$ No stat points.

But notice $y' < 0$ everywhere so curve is decreasing

pts of inflex: $y'' = 0$: $-60x^3 - 60x = 0$

$-60x(x^2 + 1) = 0$

$x = 0$, $x^2 + 1 = 0$

$x^2 = -1$

↑ No soln

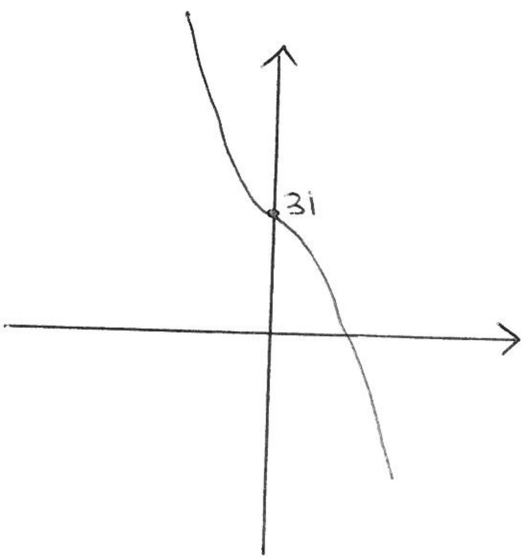
$\therefore$ $x = 0$ is a pt of inflex

when $x = 0$, $y = 31$

check concavity changes:

x	-1	0	1
y''	$+$	0	$-$
	$\cup$		$\cap$

Yes.

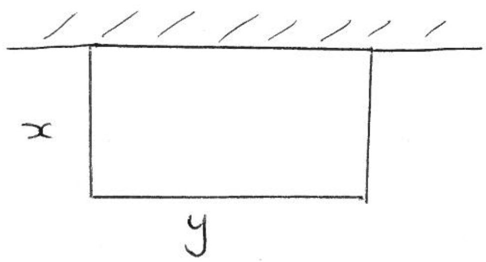


Applications - Max + Min Problems

- We can use what we know about curves to solve certain problems which ask us to find maximums and minimums.

eg

2. A rectangular paddock needs to be constructed using a straight river as one side and 200 metres of fencing material for the other 3 sides. Find the dimensions of the paddock which will maximise the area enclosed.



We want Dimension to max area.

We know perimeter : $2x + y = 200$

area : $A = xy$

Since we want to maximise area.

$$A = xy$$
$$= x(200 - 2x)$$ ← Need it to be in terms of 1 var.

$$\therefore A = 200x - 2x^2$$ ← Treat like a curve + find max stat pt.

$$\frac{dA}{dx} = 200 - 4x$$

$$\frac{d^2A}{dx^2} = -4.$$

$\therefore$ stat pt : $200 - 4x = 0$

$$4x = 200$$
$$x = 50$$

and $\frac{d^2A}{dx^2} < 0$ so this is a maximum

$\therefore$ Max area occurs when $x = 50$

$\therefore y = 200 - 2(50) = 100$

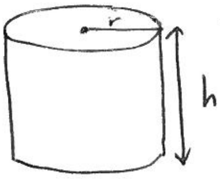
(and area is $50 \times 100 = 5000$)

Steps for max/min problems :

1. Identify what you need to max/min -ise
2. Write down what you know
3. Find a formula to be max/min - ised.
4. Find max/min pt by getting stationary points
5. Check that this is max/min.

eg.

3. A manufacturer wishes to maximise the volume of a cylindrical can that can be made out of 600π square centimetres of sheet metal. Find the dimensions of the can of greatest volume.



Want : Dimensions r , h to maximise volume

We know : $V = \pi r^2 h$

Surface area $= S = 2\pi r^2 + 2\pi r h = 600\pi$

Find formula : we want volume

$\therefore V = \pi r^2 h$ and $2\pi r^2 + 2\pi r h = 600\pi$

$2\pi r h = 600\pi - 2\pi r^2$

$$h = \frac{600\pi - 2\pi r^2}{2\pi r}$$

$$= \frac{\cancel{2\pi}(300 - r^2)}{\cancel{2\pi} r}$$

$$\therefore V = \pi r^2 \left(\frac{300 - r^2}{r} \right)$$

$$= 300\pi r - \pi r^3$$

maximise: $\frac{dV}{dr} = 300\pi - 3\pi r^2$

$$\frac{d^2V}{dr^2} = -6\pi r$$

want stat pts : $\frac{dV}{dr} = 0$: $300\pi - 3\pi r^2 = 0$

$3\pi r^2 = 300\pi$

$r^2 = 100$

$r = \pm 10$

$r = 10$ (ignore neg since r is a distance)

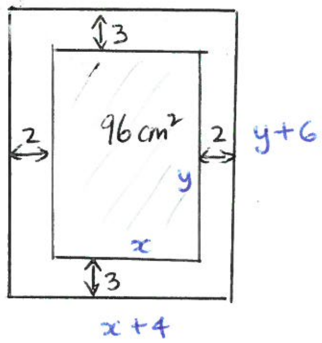
check this is max : $\frac{d^2V}{dr^2} = -6\pi r$

when $r = 10$: $\frac{d^3V}{dr^2} = -60\pi < 0$ $\cap$ max

Max Volume occurs when $r = 10$

$\therefore h = \frac{300 - r^2}{r} = \frac{300 - 10^2}{10} = 20$

4. A certain rectangular poster requires $96cm^2$ for the printed message and must have a margin around the area of print. The top and bottom margins are required to be $3cm$ and the side margins are required to be $2cm$. Find the overall dimensions of the poster if the amount of paper used is a minimum.



Let the dimensions of print be x and y

$\therefore xy = 96$

$A = (x+4)(y+6)$

We want dimensions that minimise area A.

$$A = (x+4)(y+6)$$
$$= (x+4)\left(\frac{96}{x}+6\right)$$
$$= 96 + 6x + \frac{384}{x} + 24$$
$$= 120 + 6x + 384x^{-1}$$

$$\frac{dA}{dx} = 6 - 384x^{-2}$$
$$\frac{d^2A}{dx^2} = 768x^{-3}$$

St pt: $\frac{dA}{dx} = 0 \therefore 6 - 384x^{-2} = 0$

$$\frac{384}{x^2} = 6$$
$$x^2 = \frac{384}{6}$$
$$x^2 = 64$$

$x = \pm 8$ (ignore neg) $\therefore x = 8$

check this is min $\frac{d^2A}{dx^2} = \frac{768}{8^3} > 0$ ∪ min

$\therefore$ Dimensions are $x+4 = 12$

$$y+6 = \frac{96}{x}+6 = \frac{96}{8}+6 = 18$$