11) Find the tangent and normal to the curve $y = \cos^2 x$ at the point $(\frac{\pi}{4}, \frac{1}{2})$

Tangent : $y - y_1 = m(x - x_1)$ + $(x_1, y_1) = (\frac{\pi}{4}, \frac{1}{2})$

m = slope at $x = \pi/4$
= derivative at $x = \pi/4$: $y = \cos^2 x = (\cos x)^2$

$\frac{dy}{dx} = -\sin x \cdot 2\cos x$
$= -2\sin x \cos x$
$= -\sin 2x$

$\therefore$ at $x = \frac{\pi}{4}$, $\frac{dy}{dx} = -\sin 2(\frac{\pi}{4})$
$= -\sin(\frac{\pi}{2})$
$= -1$

$\therefore m = -1$

$\therefore$ Tangent is $y - \frac{1}{2} = -1(x - \pi/4)$
$y - \frac{1}{2} = -x + \pi/4$
$y = -x + \pi/4 + \frac{1}{2}$

The normal has gradient m where $m \times -1 = -1$
$\therefore m = 1$

$\therefore$ Normal is $y - \frac{1}{2} = 1(x - \pi/4)$
$y = x - \pi/4 + \frac{1}{2}$

Curve Sketching

Recall:
- Stat pts → solve $f'(x) = 0$
- Classify them → 1st derivative test - test sign of deriv
  → 2nd deriv test - concavity
- Pts of Inflexion → Solve $f''(x) = 0$
  (+ check concavity changes)

eg12) $y = x^4 - 4x^3 - 2$

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x$

st pts: $y' = 0$: $4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0, 3$

when $x=0$: $y = -2 \longrightarrow (0,-2)$

when $x=3$: $y = 3^4 - 4(3)^3 - 2 = -29 \rightarrow (3,-29)$

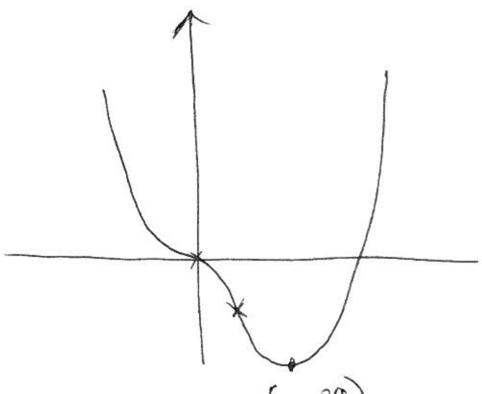
classifying: $(0,-2)$: $y'' = 0$ ∴ can't use 2nd deriv test

∴ 1st deriv test

x	-1	0	1
f'	-16	0	-8
	$-$	0	$-$

horiz pt of inflex

$(3,-29)$: $y'' = 12(3)^2 - 24(3) = 36 > 0$ ∪ min



Pts of inflex: $y'' = 0$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0,\ x = 2$

we knew about this one

eg b) $y = x + \frac{1}{x+3}$

$= x + (x+3)^{-1}$

$y' = 1 - (x+3)^{-2}$

$y'' = 2(x+3)^{-3}$

st pts : $y' = 0$: $1 - (x+3)^{-2} = 0$

$\frac{1}{(x+3)^2} = 1$

$(x+3)^2 = 1$

$x+3 = \pm 1$

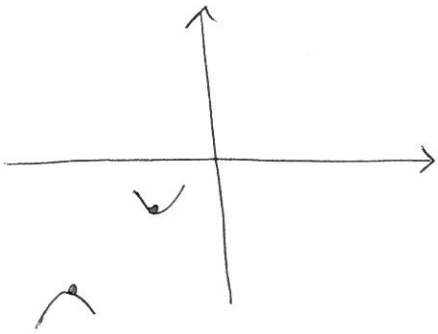
$x = -3 \pm 1 \rightarrow x = -2, -4$

when $x = -2$: $y = -2 + \frac{1}{-2+3} = -1 \rightarrow (-2, -1)$

$x = -4$: $y = -4 + \frac{1}{-4+3} = -5 \rightarrow (-4, -5)$

nature : when $x = -2$: $y'' = 2(-2+3)^{-3} = 2 > 0$ $\cup$ min

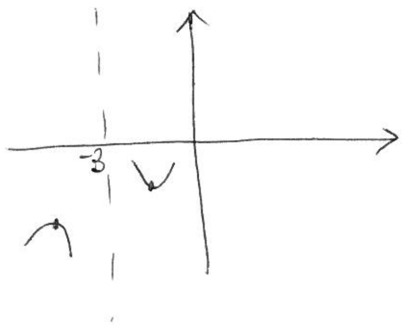
$x = -4$: $y'' = 2(-4+3)^{-3} = -2 < 0$ $\cap$ max



← Looks a bit odd.

But · Notice Domain : $x \neq -3$

∴ We have a vertical asymptote at $x = -3$



* Asymptote = Line that curve approaches but doesn't cross.

check intercepts :

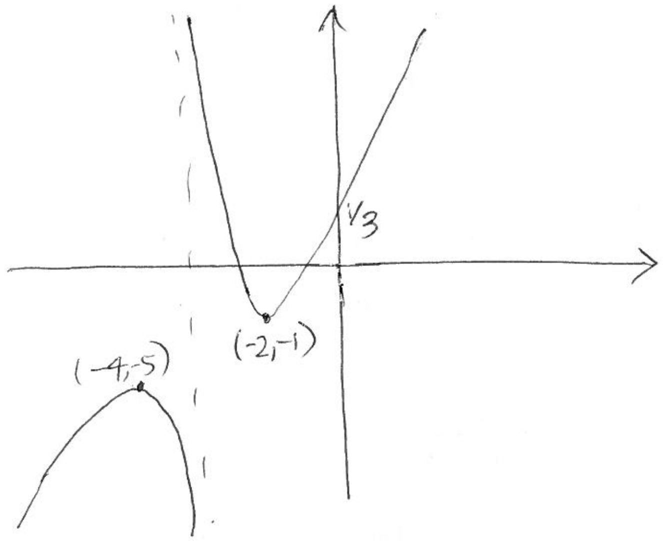
y int → Let $x=0$: $y = 0 + \frac{1}{3} = \frac{1}{3}$

x int → Let $y=0$: $x + \frac{1}{x+3} = 0$

$$x(x+3) + 1 = 0$$

$$x^2 + 3x + 1 = 0$$

$$x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$



Other things to look for :

- Notice there are no points of inflexion $y'' = 0 \rightarrow \frac{2}{(x+3)^3} = 0$

$2 = 0$ No soln.

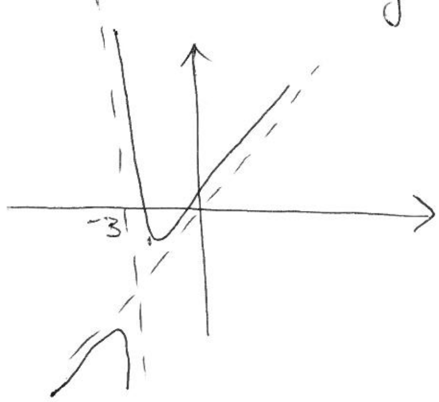
- Look at behaviour

as $x \to \infty$, $y = x + \frac{1}{x+3}$

↖ this gets smaller

$\therefore y \to \infty$ along the line $y = x$

↗ oblique asymptote



Curve Sketching:

1. Stat pts → Solve $f'(x) = 0$

2. Find their nature → 1st deriv test
→ 2nd deriv test

3. Pts of Inflexion → Solve $f''(x) = 0$ (+ check concavity changes)

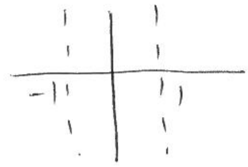
4. Intercepts

5. Behaviour → what happens as $x \to \infty$

6. Asymptotes
↑ Line function approaches
. vertical asymptotes → check domain
· horizontal or oblique asymptotes → Look at $x \to \pm\infty$

eg 2c) $y = \dfrac{1}{1-x^2}$

Note: $x \neq \pm 1$ → vertical asymptotes



$y = (1-x^2)^{-1}$

$y' = -(1-x^2)^{-2} \cdot (-2x)$

$= \dfrac{2x}{(1-x^2)^2}$

St pts : $y' = 0$ ie: $\dfrac{2x}{(1-x^2)^2} = 0$

ie: $2x = 0$

$x = 0$

when $x = 0$: $y = \dfrac{1}{1-0^2} = 1$ → $(0, 1)$

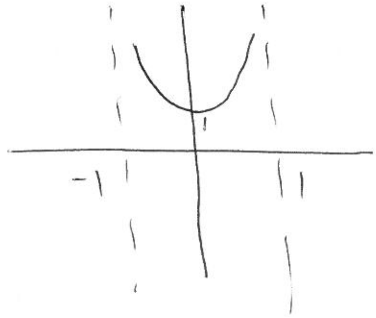
nature : use 1st deriv test

at $(0,1)$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$\frac{dy}{dx}$	$-$	0	$+$
	\	—	/

$$\frac{dy}{dx} = \frac{2x}{(1-x^2)^2} \leftarrow \text{Pos}$$

min at $(0,1)$



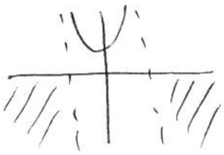
• Intercepts : $y\text{int} \to x=0$: $y = \frac{1}{1-0} = 1$ (we knew this)

$x\text{int} \to y=0$: $\frac{1}{1-x^2} = 0$

$1 = 0$ No sense
No soln
No x-intercepts.

• Behaviour :

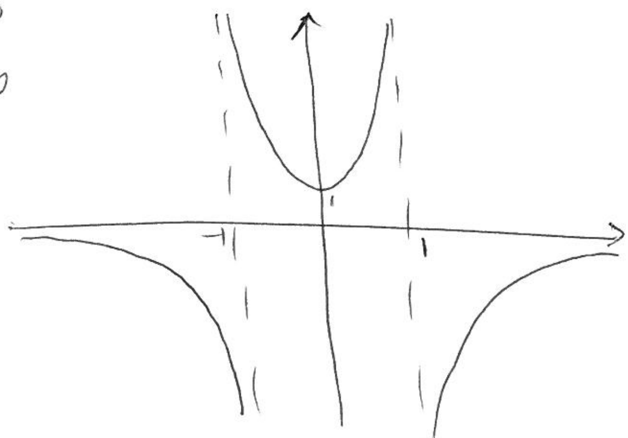
Notice for $x>1$, y is neg
$x<-1$, y is neg



As $x \to \infty$, $y = \frac{1}{1-x^2} \to 0$

$x \to -\infty$, $y \to 0$

$x \to \pm 1$, $y \to \infty$



eg 12d) $y = xe^x$

$y' = xe^x + e^x$ (prod rule)

$y'' = xe^x + e^x + e^x$

$= xe^x + 2e^x$

st pts : $y' = 0$: $xe^x + e^x = 0$

$e^x(x+1) = 0$

$\therefore x + 1 = 0$

$x = -1$

when $x = -1$, $y = -e^{-1}$ $\rightarrow \left(-1, -\frac{1}{e}\right)$

nature : when $x = -1$: $y'' = -e^{-1} + 2e^{-1} = e^{-1} > 0$ $\cup$ min

pts of inflex : $y'' = 0$: $xe^x + 2e^x = 0$

$e^x(x+2) = 0$

$x = -2$

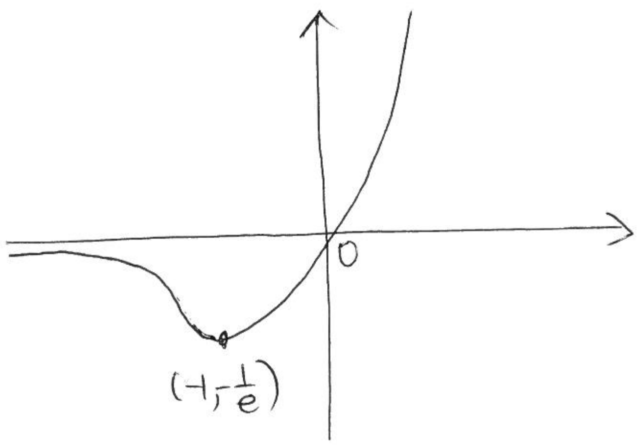
check concavity changes

x	-3	-2	-1
y''	$-e^{-3}$	0	$\cup$
	$\cap$		↑ we know this

✓

Intercepts : y int $\rightarrow$ $x = 0$: $y = 0e^0 = 0$

x int $\rightarrow$ $y = 0$: $xe^x = 0 \rightarrow x = 0$



Behaviour

as $x \rightarrow \infty$, $y = xe^x \rightarrow \infty$

as $x \rightarrow -\infty$: $y = xe^x \rightarrow 0$

↑ dominates + gets very small.