Derivatives of Special Functions

So Far we have learnt how to differentiate powers of x and polynomials

ie: $\frac{d}{dx}(x^n) = nx^{n-1}$

We can now expand our knowledge of differentiation to find derivatives of other functions we know about.

ie: Trig functions → sin, cos, tan

Exp functions

Log functions

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\log x$	$\frac{1}{x}$

Proofs: To prove these derivatives we must use First Principles. At this stage our knowledge of limits is slightly lacking to fully understand the proofs.

But heres one:

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h \to 0} \sin x \left(\frac{\cosh - 1}{h}\right) + \cos x \left(\frac{\sinh}{h}\right)$$

$$= \sin x \underbrace{\left[\lim_{h \to 0} \frac{\cosh - 1}{h}\right]}_{\downarrow\, 0} + \cos x \underbrace{\left[\lim_{h \to 0} \frac{\sinh}{h}\right]}_{\downarrow\, 1}$$

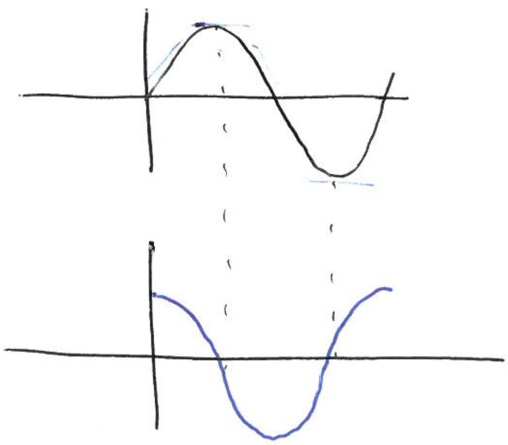
$$= \cos x$$

The reason we work in radians is this.

we can probably understand this more from the graph.

$f(x) = \sin x$:

Sketching $f'(x)$



$f'(x) = \cos x$

See link on ilearn for other proofs

Egs

Differentiate the following functions:

4 a) $y = e^x + x^5$

$$\frac{dy}{dx} = e^x + 5x^4$$

b) $y = \sin x + \cos x$

$$y' = \cos x - \sin x$$

c) $y = 3\log x - 2e^x + 4\tan x$

$$\frac{dy}{dx} = 3\left(\frac{1}{x}\right) - 2e^x + 4\sec^2 x$$

$$= \frac{3}{x} - 2e^x + 4\sec^2 x$$

Eg5) Use the Product Rule to differentiate the following

a) $f(x) = x^2 e^x$

Let $u = x^2$ $\quad v = e^x$

$u' = 2x$ $\quad v' = e^x$

$$f'(x) = x^2 e^x + e^x(2x)$$

$$= x^2 e^x + 2x e^x$$

b) $g(x) = x \ln x$

$u = x$ $\quad v = \ln x$

$u' = 1$ $\quad v' = \frac{1}{x}$

$$g'(x) = x\left(\frac{1}{x}\right) + \ln x(1)$$

$$= 1 + \ln x$$

c) $h(x) = e^x \sin x$

$u = e^x \qquad v = \sin x$

$u' = e^x \qquad v' = \cos x$

$h'(x) = e^x(\cos x) + \sin x\,(e^x)$

$= e^x \cos x + e^x \sin x$

6) use the Quotient Rule to differentiate the following

a) $f(x) = \dfrac{e^x}{1+x}$

$u = e^x \qquad v = 1+x$

$u' = e^x \qquad v' = 1$

$$f'(x) = \frac{(1+x)e^x - e^x(1)}{(1+x)^2}$$

$$= \frac{e^x(1+x-1)}{(1+x)^2}$$

$$= \frac{xe^x}{(1+x)^2}$$

b) $g(x) = \dfrac{2\sqrt{x}}{\ln x}$

$u = 2\sqrt{x} = 2x^{1/2} \qquad v = \ln x$

$u' = x^{-1/2} \qquad v' = 1/x$

$$g'(x) = \frac{\ln x\,(x^{-1/2}) - 2x^{1/2}\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$= \frac{x^{-1/2}\ln x - 2x^{-1/2}}{(\ln x)^2}$$

Eg: Use the chain rule to differentiate the following

7a) $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

deriv of inside

deriv of main function

Notice: $y = e^{ax}$

$\frac{dy}{dx} = ae^{ax}$

b) $y = e^{10x}$

$y' = 10e^{10x}$

c) $y = e^{x^2}$

$\frac{dy}{dx} = 2x\,e^{x^2}$

d) $y = e^{x^3+4x}$

$\frac{dy}{dx} = (3x^2+4)\,e^{x^3+4x}$

e) $y = \sin 4x$

$y' = 4\cos 4x$

Notice: $y = \sin ax$

$\frac{dy}{dx} = a\cos ax$

f) $y = \sin(2-3x)$

$\frac{dy}{dx} = -3\cos(2-3x)$

g) $y = \cos(x^2)$

$$\frac{dy}{dx} = 2x\,(-\sin(x^2))$$

$$= -2x\sin(x^2)$$

h) $y = \tan(5x+4)$

$y' = 5\sec^2(5x+4)$

i) $y = \log(3x+1)$

$$\frac{dy}{dx} = 3\left(\frac{1}{3x+1}\right) = \frac{3}{3x+1}$$

j) $y = \log(x^2+4)$

$$y' = 2x\left(\frac{1}{x^2+4}\right) = \frac{2x}{x^2+4}$$

k) $y = \log(\sin x)$

$$\frac{dy}{dx} = \cos x \cdot \frac{1}{\sin x} = \frac{\cos x}{\sin x} = \cot x$$

l) $y = (\log x)^3$

$y' = \frac{1}{x} \cdot 3(\log x)^2 = \frac{3(\log x)^2}{x}$

m) $y = \sqrt{e^x + 2}$

$= (e^x+2)^{1/2}$

$\frac{dy}{dx} = e^x \cdot \frac{1}{2}(e^x+2)^{-1/2}$

$= \frac{e^x}{2}(e^x+2)^{-1/2}$

n) $y = 3\cos\sqrt{x}$

$\frac{dy}{dx} = 3\left(\frac{1}{2}x^{-1/2} \cdot (-\sin\sqrt{x})\right)$

$= \frac{-3}{2}x^{-1/2}\sin\sqrt{x}$

Try Q8

eg9) a) $y = \sin^2 x + \sin(x^2)$

$= (\sin x)^2 + \sin(x^2)$

Lets do these separately + then add together

$f = (\sin x)^2$ (chain rule)

$f' = \cos x \,.\, 2(\quad)'$

$= \cos x \cdot 2\sin x$

$= 2\sin x \cos x$

$= \sin 2x.$

and $g = \sin(x^2)$

$\therefore g' = 2x \cdot \cos(\quad)$

$= 2x\cos(x^2)$

$\therefore \dfrac{dy}{dx} = \sin 2x + 2x\cos(x^2)$

eg 9b) $y = \sin(x \log x)$

Let $u = x\log x \rightarrow \frac{du}{dx} = x\left(\frac{1}{x}\right) + \log x\,(1) = 1 + \log x$

$\therefore y = \sin u \rightarrow \frac{dy}{du} = \cos u$

$$\therefore \frac{dy}{dx} = (1 + \log x)\cdot \cos u$$
$$= (1 + \log x)\cdot \cos(x \log x)$$

eg 9c) $y = \log(xe^{-x})$

Let $u = xe^{-x} \rightarrow \frac{du}{dx} = x(-e^{-x}) + e^{-x}(1) = -xe^{-x} + e^{-x}$

$y = \log u \Rightarrow \frac{dy}{du} = \frac{1}{u}$

$$\therefore \frac{dy}{dx} = (-xe^{-x} + e^{-x})\,\frac{1}{u}$$
$$= \frac{-xe^{-x} + e^{-x}}{xe^{-x}}$$
$$= \frac{\cancel{e^{-x}}(-x+1)}{x\cancel{e^{-x}}}$$
$$= \frac{1-x}{x}$$

Differentiating other exponentials.

eg: $y = 2^x$

Rearrange: $y = 2^x$

$\log y = \log 2^x$

$\log y = x \log 2$

$\therefore y = e^{x \log 2}$

$\therefore$ using chain rule:

Let $u = x \log 2 \rightarrow \frac{du}{dx} = \log 2$

$\therefore y = e^u \longrightarrow \frac{dy}{du} = e^u$

$\therefore \frac{dy}{dx} = \log 2 \cdot e^u$

$= \log 2 \cdot e^{x \log 2}$

$= \log 2 \cdot 2^x$

Infact: $y = a^x$

then $\frac{dy}{dx} = \log a \cdot a^x$