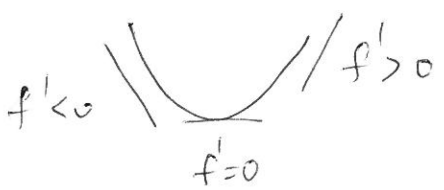
Differentiation + Curve Sketching

So Far:

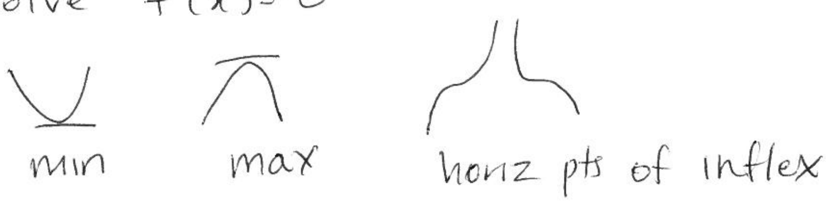
- Derivative = slope of tangent
  = rate of change
- First principles
- Product, Quotient, Chain
- Tangents + Normals

- Using the first derivative



If $f' > 0 \Rightarrow f(x)$ increasing
$f' < 0 \Rightarrow f(x)$ decreasing
$f' = 0 \Rightarrow f(x)$ stationary

Stationary points $\rightarrow$ solve $f'(x) = 0$



- 1st derivative test can tell us their nature

x		st pt	
y'	+	0	−
	/	—	\

Second Derivative

= derivatve of the derivative

eg: $y = 10x^3 + 7x$

$y' = 30x^2 + 7$

$y'' = 60x$

denoted by $y'' = f''(x) = f^{(2)}(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

eg i) Find the second derivatve of $f(x) = (x^2+1)^4$

$f'(x) = 2x \cdot 4(x^2+1)^3$ (chain Rule)

$= 8x(x^2+1)^3$

$f''(x)$: Use Product rule : $u = 8x$, $u' = 8$; $v = (x^2+1)^3$, $v' = 2x \cdot 3(x^2+1)^2 = 6x(x^2+1)^2$

$\therefore f''(x) = uv' + vu'$

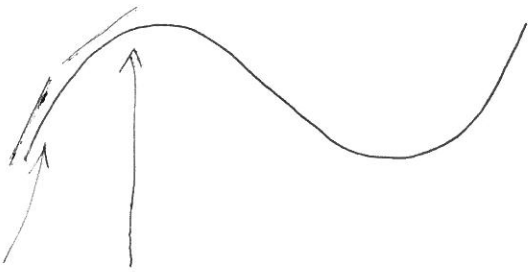
$= 8x \cdot 6x(x^2+1)^2 + (x^2+1)^3 \cdot 8$

$= 8(x^2+1)^2 \left[6x^2 + (x^2+1)\right]$

$= 8(x^2+1)^2(7x^2+1)$

1st derivative = slope of tangent

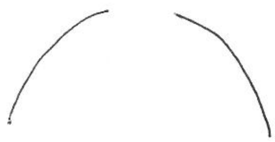
2nd derivative = change in this slope



steep $\qquad$ less steep

y' $\quad$ big pos ..... small pos n°

y' $\quad$ decreasing $\Rightarrow y'' < 0$



similarly

$y'' < 0$ $\qquad\qquad$ $y'' > 0$

y'' tells us about the concavity

concave down

Concave up

notice $\quad$ Max = concave down $(y''<0)$, min = concave up $(y''>0)$

$\therefore$ We can use the 2nd derivative to tell us the nature of some of our stat pts.

2nd derivative test:

If $f''>0 \rightarrow$ concave up $\cup$ minimum

$f''<0 \rightarrow$ concave down $\cap$ maximum

$f''=0 \rightarrow$ no infor about stat pt

eg 2a) $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$

stat pts: $f' = 0$

$3x^2 - 12 = 0$

$x^2 = 4$

$x = \pm 2$

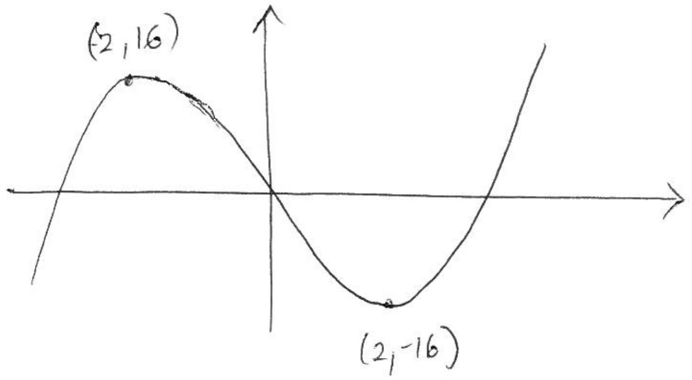
when $x = 2$, $y = f(2) = 2^3 - 12(2) = -16 \rightarrow (2,-16)$

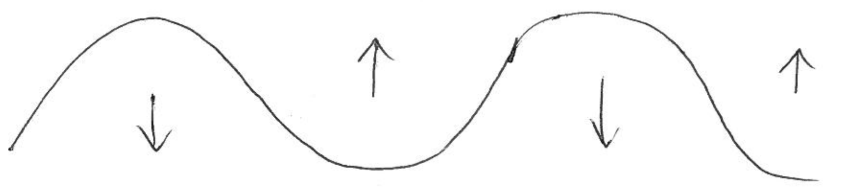
$x = -2$, $y = f(-2) = 16 \rightarrow (-2, 16)$

classifying st pts:

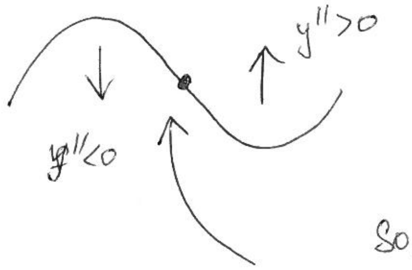
$(2,-16)$ $f''(x) = 6(2) = 12 > 0$ $\cup$ min

$(-2, 16)$ $f''(x) = 6(-2) = -12 < 0$ $\cap$ max





y'' $-$ $+$ $-$ $+$



Somewhere inbetween y'' changes from $+$ to $-$.

ie: concavity changes from ↑ to ↓ or ↓ to ↑

y'' changes from $+$ to $-$ or $-$ to $+$

ie: $y''=0$ at these points

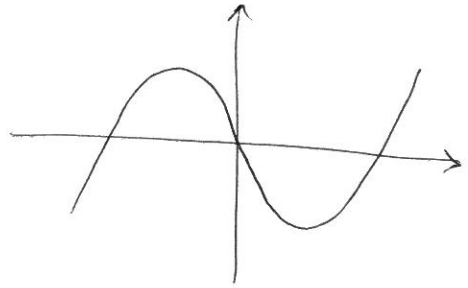
These points where concavity changes are called points of inflexion

- Points of Inflexion = point where concavity changes
- Finding points of inflexion:
  1. Find where $y''=0$
  2. Test that concavity changes

Back to eg 2b) $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x$



Points of inflex : $f'' = 0$

ie: $6x = 0$

ie: $x = 0$

test concavity does change

x	-1	0	1
f''	-6	0	6

Note: Since we already know this lies between 2 stat pts, then we can already see concavity changes.

So here pt of inflex occurs at $x = 0$ (and $y = f(0) = 0^3 - 12(0)$)

ie: pt of inflex is $(0, 0)$

eg: $y = x^4$

$y' = 4x^3$

$y'' = 12x^2$

Pts of inflex : $12x^2 = 0$

$x = 0$

test concavity:

x	-1	0	1
y''	12	0	12
	$+$		$+$

No change

$\therefore$ Not a pt of inflexion

(Infact this is a curve with a flat bottom)

Sketching Curves

1. Find stat points
2. Classify them (1st deriv test or 2nd deriv test)
3. Points of Inflexion
3. Other details - intercepts, behaviour etc

eg3a) $y = x^4 - 8x^2 + 16$

$y' = 4x^3 - 16x$

$y'' = 12x^2 - 16$

st pts: $y' = 0$: $4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$x = 0, 2, -2$

when $x = 0$: $y = 16 \rightarrow (0,16)$

$x = \pm 2$: $y = 2^4 - 8(2)^2 + 16 = 0 \rightarrow (2,0), (-2,0)$

nature: $(0,16)$ $y'' = -16 < 0$ $\cap$ max

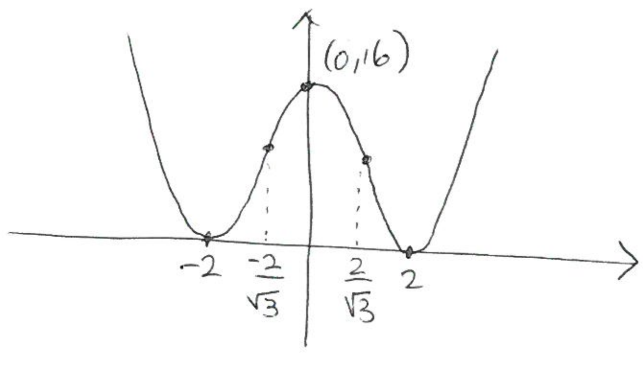
$(2,0)$ $y'' = 12(2)^2 - 16 > 0$ $\cup$ min

$(-2,0)$ $y'' = 32 > 0$ $\cup$ min

pts of inflex: $y'' = 0$ ie: $12x^2 - 16 = 0$

$x^2 = \frac{16}{12} = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}$



3b) $y = x^4 - 2x^3 + 2$

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x$

st pts : $4x^3 - 6x^2 = 0$

$2x^2(2x - 3) = 0$

$x = 0, \, 3/2$

when $x = 0 : y = 0 - 0 + 2 \rightarrow (0, 2)$

$x = \frac{3}{2} : y = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 + 2 = \frac{5}{16} \rightarrow \left(\frac{3}{2}, \frac{5}{16}\right)$

nature : $\left(\frac{3}{2}, \frac{5}{16}\right)$: (2nd deriv test)

$y'' = 27 - 18 = 9 > 0$ $\cup$ min

$(0, 2)$: $y'' = 12 - 12 = 0$ ← No info about concavity! can't use 2nd deriv test

using 1st deriv test

x	-1	0	1
y'	-10	0	-2
	$-$	0	$-$

∴ horizontal pt of inflex

pt of inflex : $12x^2 - 12x = 0$

$12x(x - 1) = 0$

$x = 0, \, 1$

we knew about this one.