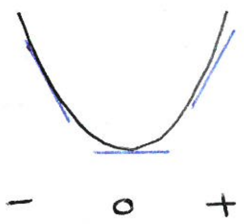
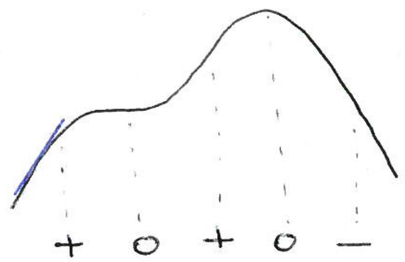
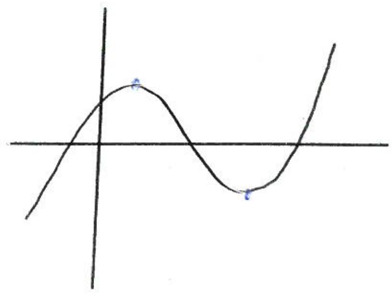
Derivatives and Functions

- Derivative = slope of tangent to curve.
- We saw: If $f' > 0 \Rightarrow f$ increasing
  $f' < 0 \Rightarrow f$ decreasing
  $f' = 0 \Rightarrow f$ stationary

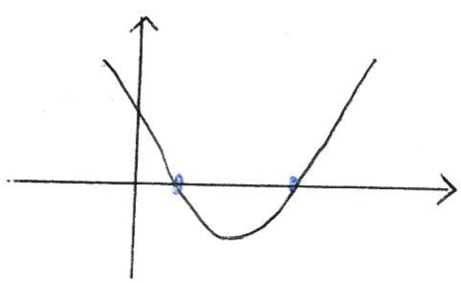




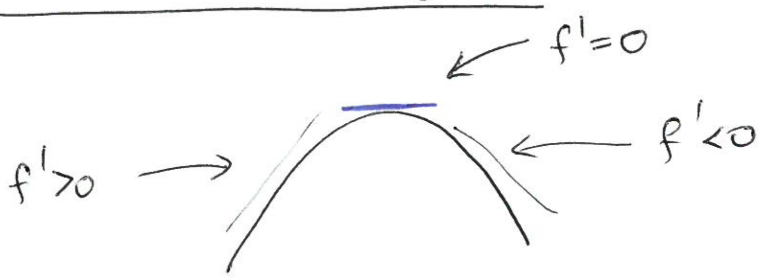
Suppose $f(x)$ looks like:



Then we can sketch $f'(x)$ by looking at the slope:



What about Stationary Points



- The stationary point gives us the point where our function turns.

- Stationary point = point where slope is 0
  = point where derivative is 0

- 3 types of stat pts :

local maximum | local minimum | horizontal points of inflexion

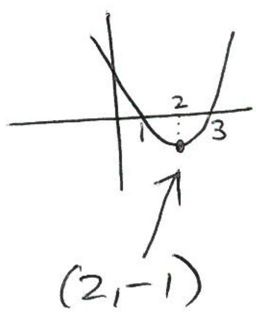
$\underbrace{\text{local maximum} \quad \text{local minimum}}_{\text{turning points}}$

- Finding stat pts $\Rightarrow$ want x values where $f'(x)=0$

$\therefore$ $\boxed{\text{Finding stat pts} \Rightarrow \text{Solve } f'(x)=0}$

eg: $f(x) = x^2-4x+3$. Find any stationary points.

$f'(x)=2x-4$ + we want
$$2x-4=0$$
$$2x=4$$
$$x=2$$



when $x=2$: $y=f(2)=2^2-4(2)+3=-1$

eg) Find the stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$.

stat pts $\Rightarrow$ Solve $f'(x) = 0$

$\therefore f'(x) = 3x^2 - 6x - 9$

So $3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$\therefore x^2 - 2x - 3 = 0$

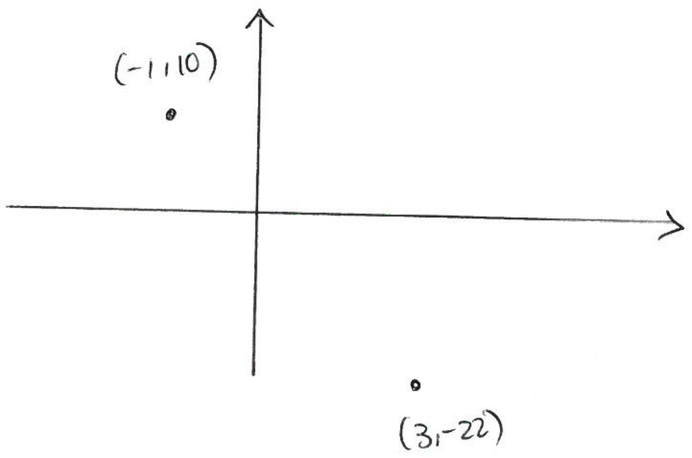
$(x-3)(x+1) = 0$

$x = 3, -1$

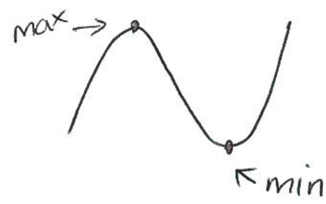
when $x=3$: $y = f(3) = 3^3 - 3(3)^2 - 9(3) + 5 = -22 \rightarrow (3,-22)$

when $x=-1$: $y = f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = -10 \rightarrow (-1,10)$

$\therefore$ There are 2 stationary points $(3,-22)$ + $(-1,10)$



note : since $f(x)$ is a cubic we know what this will look like!

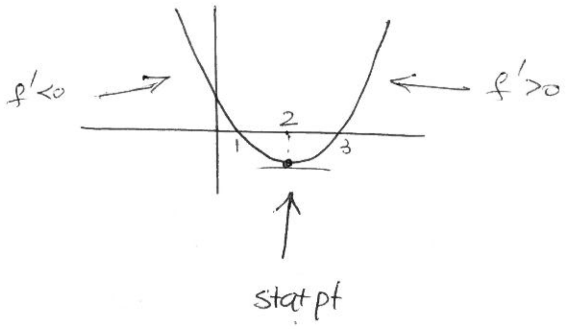


But what if we didn't know what our curve looked like

How do we find the type of stat point?

Back to eg: $f(x) = x^2 - 4x + 3$

We saw



$-$ 0 $+$

∴ Looking at slope on either side of stat pt

$-$ 0 $+$ | $+$ 0 $-$ | $+$ 0 $+$ | $-$ 0 $-$

min | max | |

horiz pts of inflexion

1st derivative test → test slope on either side of stationary point

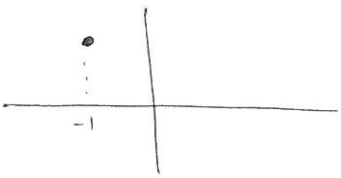
- this will tell us the 'nature' of the stat point eg $\quad f(x) = x^3 - 3x^2 - 9x + 5$

$f'(x) = 3x^2 - 6x - 9$

we found stat points $(-1, 10)$ and $(3, -22)$

Find their nature → use 1st derivative test

$(-1, 10)$



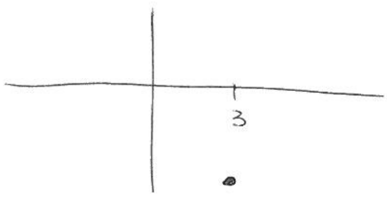
x	-2	-1	0
$\frac{dy}{dx}$	35	0	-9

$3(-2)^2 - 6(-2) - 9 \qquad 3(0)^2 - 6(0) - 9$

$+ \quad 0 \quad -$

max

$(3, -22)$



x	2	3	4
$\frac{dy}{dx}$	-9	0	3

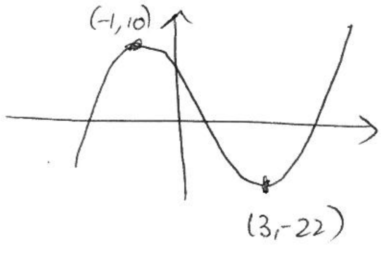
$3(2)^2 - 6(2) - 9 \qquad 3(4)^2 - 6(4) - 9$

$- \quad 0 \quad +$

min

$\therefore (-1, 10)$ is a max + $(3, -22)$ is a min

our graph must look like



(Remember polys are continuous
ie: no breaks)

$f(x) = 3x^4 - 4x^3 + 5$. Find the stationary points and classify them.

$f(x) = 3x^4 - 4x^3 + 5$

$f'(x) = 12x^3 - 12x^2$

stat pts $\rightarrow f' = 0$ ie: $12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$\therefore x = 0, 1$

when $x = 0$: $y = f(0) = 3(0)^4 - 4(0)^3 + 5 = 5 \rightarrow (0,5)$

$x = 1$: $y = f(1) = 3(1)^4 - 4(1) + 5 = 4 \rightarrow (1,4)$

classifying :

at $(0,5)$: 1st deriv test

x	-1	0	$\frac{1}{2}$
f'	-24	0	$-$

can't pick 1 as thats the other st pt.

$12(\frac{1}{2})^3 - 12(\frac{1}{2})^2$

$= \frac{12}{8} - \frac{12}{4}$

$-\ 0\ -$

horiz pt of inflex

at $(1,4)$: 1st deriv test

x	$\frac{1}{2}$	1	2
f'	$-$	0	$+$

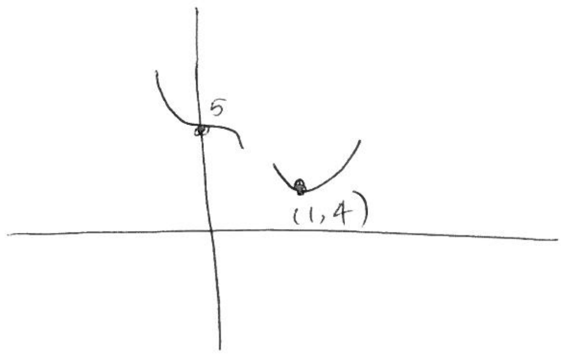
$12(2)^3 - 12(2)^2$

min

$\therefore (0,5)$ is a horiz pt of inflex

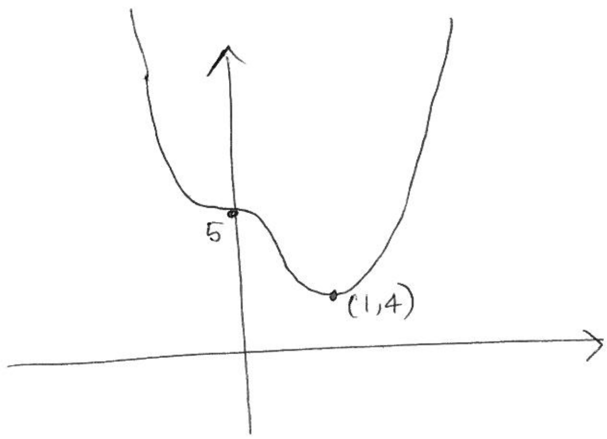
$(1,4)$ is a min

We can picture whats happenning:



Note: - This is a polynomial so it's continuous
- There are no breaks
- no problems with domain so $Dom = \mathbb{R}$

$\therefore$ This must look like:



Eg: $y = x^3 - 12x$

$$\frac{dy}{dx} = 3x^2 - 12$$

Stat pts: $\frac{dy}{dx} = 0$

$3x^2 - 12 = 0$

$x^2 = 4$

$x = \pm 2$

when $x = 2$: $y = 8 - 24 = -16 \rightarrow (2,-16)$

$x = -2$: $y = -8 + 24 = 16 \rightarrow (-2,16)$

Classifying: at $(2,-16)$

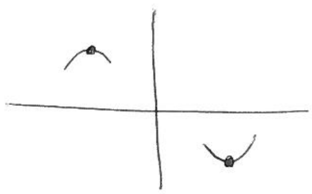
x	1	2	3
$\frac{dy}{dx}$	-9	0	15
	$-$	0	$+$
	\	—	/

min

$(-2, 16)$

x	-3	-2	-1
$\frac{dy}{dx}$	15	0	-9
	$+$	0	$-$
	/	—	\

max



More details: - Dom = $\mathbb{R}$ & this is continuous ∴ joins up nicely

- Intercepts: yint → Let $x=0$: $y = 0 - 0 = 0$

x int → Let $y=0$: $x^3 - 12x = 0$

$x(x^2 - 12) = 0$

$x = 0, \pm\sqrt{12}$