Yesterday

- Extending our techniques of differentiation

Product Rule $y = uv$

$$\frac{dy}{dx} = uv' + vu'$$

Quotient Rule $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

Chain Rule: $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

↗

Look for function inside a function

eg: $y = (5x^2 - 6)^8$

Let $u = 5x^2 - 6 \rightarrow \frac{du}{dx} = 10x$

$y = u^8 \rightarrow \frac{dy}{du} = 8u^7$

$$\therefore \frac{dy}{dx} = 10x \times 8u^7$$

$$= 80x(5x^2-6)^7$$

eg $\quad y = (4x^3 + 6x^2 - 3x)^{10}$

Let $\quad u = 4x^3 + 6x^2 - 3x \rightarrow \dfrac{du}{dx} = 12x^2 + 12x - 3$

$\therefore \quad y = u^{10} \longrightarrow \dfrac{dy}{du} = 10u^9$

$$\therefore \frac{dy}{dx} = 10u^9 \times (12x^2 + 12x + 3)$$

$$= 10(12x^2 + 12x + 3)(4x^3 + 6x^2 - 3x)^9$$

eg $\quad y = (x - 2x^3)^4$

Let $\quad u = x - 2x^3 \rightarrow \dfrac{du}{dx} = 1 - 6x^2$

$\therefore \quad y = u^4 \rightarrow \dfrac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = (1-6x^2) \cdot 4u^3 \quad = (1-6x^2)\,4(x-2x^3)^3$$

$$= 4(1-6x^2)(x-2x^3)^3$$

Quick method:

Notice $\dfrac{dy}{dx} = \underbrace{(1-6x^2)}_{\text{deriv of inside function}}\ \underbrace{4(\quad)^3}_{\text{deriv of main function}}$

$$= (1-6x^2)\,4(x-2x^3)^3$$

eg $y = (x^2+3)^3$

Quick method → This is x^2+3 inside the cubed function

$\therefore \frac{dy}{dx} = 2x \times 3(\quad)^2$

deriv of inside

deriv of main fn

$= 2x \times 3(x^2+3)^2$

$= 6x(x^2+3)^2$

eg $y = \sqrt{x^3+x+1}$

$= (x^3+x+1)^{1/2}$

$\frac{dy}{dx} = (3x^2+1) \times \frac{1}{2}(\quad)^{-1/2}$

$= (3x^2+1) \cdot \frac{1}{2}(x^3+x+1)^{-1/2}$

$= \frac{1}{2}(3x^2+1)(x^3+x+1)^{-1/2}$

The chain Rule is sometimes given as

$y = f(g(x))$

Then $\frac{dy}{dx} = g'(x)\, f'(g(x))$

eg $y = (4x^3 - 7x)^{1/3}$

$$\frac{dy}{dx} = (12x^2 - 7) \times \frac{1}{3}(\quad)^{-2/3}$$

$$= (12x^2 - 7) \cdot \frac{1}{3}(4x^3 - 7x)^{1/3}$$

$$= \frac{1}{3}(12x^2 - 7)(4x^3 - 7x)^{1/3}$$

eg $y = \dfrac{1}{x^4 - x^2}$

$$= (x^4 - x^2)^{-1}$$

$$\frac{dy}{dx} = (4x^3 - 2x^2) \times -(x^4 - x^2)^{-2}$$

$$= -(4x^3 - 2x^2)(x^4 - x^2)^{-2}$$

eg $y = \dfrac{1}{\sqrt{x^2+9}}$

$$= (x^2+9)^{-1/2}$$

Let $u = x^2 + 9 \longrightarrow \dfrac{du}{dx} = 2x$

$\therefore y = u^{-1/2} \longrightarrow \dfrac{dy}{du} = -\dfrac{1}{2}u^{-3/2}$

$$\therefore \frac{dy}{dx} = 2x\left(\frac{-1}{2}u^{-3/2}\right)$$

$$= -x(x^2+9)^{-3/2}$$

Try Q3

Eg4) Differentiate $f(x) = \dfrac{1}{\sqrt{(x^3+6x)(x^2+4)}}$

Product, Quotient or Chain ??

Rewrite $f(x) = \left[(x^3+6x)(x^2+4)\right]^{-1/2}$

This is a chain rule (with a product inside).

Let $u = (x^3+6x)(x^2+4)$

$\therefore$ $f = u^{-1/2} \longrightarrow \dfrac{df}{du} = -\dfrac{1}{2}u^{-3/2}$

$$\frac{du}{dx} = (x^3+6x)\,2x + (x^2+4)(3x^2+6)$$

$$= 2x^4 + 12x^2 + 3x^4 + 6x^2 + 12x^2 + 24$$

$$= 5x^4 + 30x^2 + 24$$

$$\therefore \frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx}$$

$$= -\frac{1}{2}u^{-3/2}\left(5x^4 + 30x^2 + 24\right)$$

$$= -\frac{1}{2}\left((x^3+6x)(x^2+4)\right)^{-3/2} \cdot \left(5x^4 + 30x^2 + 24\right)$$

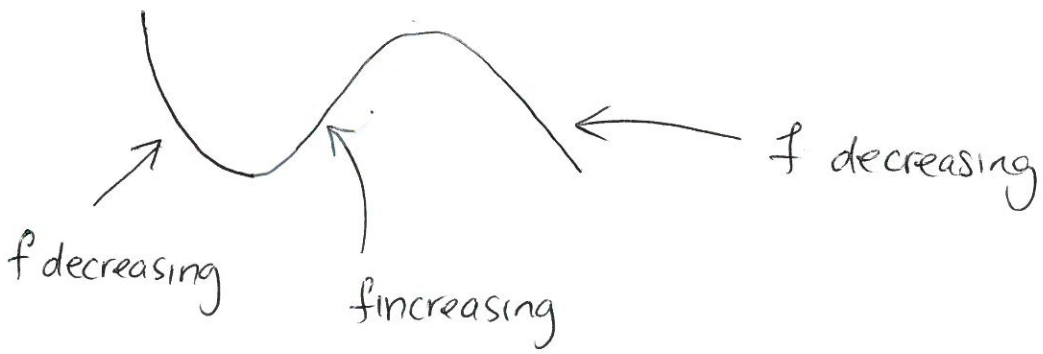
Derivatives and Functions

- The derivative can tell us a lot about our function
- Recall Derivative = rate of change

  = slope of tangent to curve

Increasing + Decreasing Functions

Recall



We said $f(x_1) \leq f(x_2)$ for $x_1 < x_2$ $\Rightarrow$ f increasing

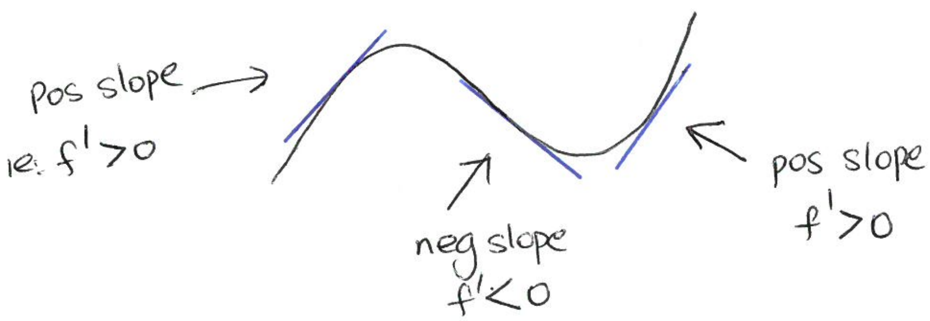
$f(x_1) \geq f(x_2)$ for $x_1 < x_2$ $\Rightarrow$ f decreasing

We now have another method of seeing when our function is increasing / decreasing

Look at slope of tangent

"

derivative



ie: + 0 − 0 +

The derivative tells us:

If $f'(x) > 0$ then $f(x)$ is increasing

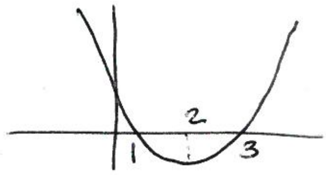
$f'(x) < 0$ then $f(x)$ is decreasing

$f'(x) = 0$ then $f(x)$ is stationary

eg: $f(x) = x^2 - 4x + 3$. Find the intervals on which $f(x)$ is increasing and decreasing.

Notice $f(x) = x^2 - 4x + 3$ ← parabola

$= (x-1)(x-3)$



↗ We can see our solution from the graph

But lets use derivatives to confirm this:

$f'(x) = 2x - 4$

Increasing: want $2x - 4 > 0$

$2x > 4$

$x > 2$

$\therefore$ Inc on $(2, \infty)$

Decreasing: want $2x - 4 < 0$

$x < 2$

Dec on $(-\infty, 2)$

eg Find the intervals on which $f(x) = 5+3x^2-x^3$ is strictly increasing and strictly decreasing.

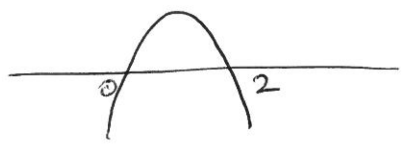
Look at $f'(x) = 6x - 3x^2$

We know $f(x)$ is strictly inc when $f'(x) > 0$

ie: $6x - 3x^2 > 0$

Quadratic inequality

Look at $y = 6x - 3x^2$
$= 3x(2-x)$



This is positive on: $0 < x < 2$

$\therefore f'(x) > 0$ on $(0,2)$ $\therefore f$ in strictly inc on $(0,2)$

Also we can see $f'(x) < 0$ on $(-\infty, 0)$ and $(2, \infty)$

$\therefore f(x)$ is strictly decreasing on $(-\infty,0) + (2,\infty)$

eg Prove $y = (2x-1)^3$ is an increasing function.

Look at $\frac{dy}{dx} = 2 \cdot 3(2x-1)^2$ (chain rule)
$= 6(2x-1)^2$

For any value of x, $\frac{dy}{dx} \geq 0$

$\therefore y$ is increasing for $x \in \mathbb{R}$