More Differentiation

Summary so far

- Derivative = slope of tangent to curve

  = rate of change

- First principles $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

- Differentiation Rules :

$f(x)$	$f'(x)$
x^n	nx^{n-1}
k	0
$k\,f(x)$	$k\,f'(x)$
$f \pm g$	$f' \pm g'$

eg i) $f(x) = 5x^2$

$f'(x) = 10x$

2) $g(t) = \sqrt{t} + 4$

$= t^{1/2} + 4$

$\frac{dg}{dt} = \frac{1}{2} t^{-1/2}$

3) $h(x) = \frac{1}{x^6} + \sqrt{3x^3}$

$= x^{-6} + \sqrt{3}\,x^{3/2}$

$\frac{dh}{dx} = (-6x^{-7}) + \sqrt{3}\left(\frac{3}{2} x^{1/2}\right)$

$= -\frac{6}{x^7} + \frac{3\sqrt{3}}{2} x^{1/2}$

Product Rule

$$\boxed{\begin{aligned} &\text{If} \quad y = u(x)\cdot v(x) \\ &\text{Then} \quad \frac{dy}{dx} = u(x)\cdot v'(x) + v(x)\,u'(x) \end{aligned}}$$

Proof: From First principles: Let $y = u(x)\cdot v(x)$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{u(x+h)\,v(x+h) - u(x)\,v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)\,v(x+h) - u(x)\,v(x+h) + u(x)\,v(x+h) - u(x)\,v(x)}{h}$$

$$= \lim_{h\to 0} \frac{v(x+h)\left[u(x+h) - u(x)\right] + u(x)\left[v(x+h) - v(x)\right]}{h}$$

$$= \lim_{h\to 0} \underset{\substack{\downarrow \\ v(x)}}{v(x+h)} \underset{\substack{\downarrow \\ u'(x)}}{\left[\frac{u(x+h) - u(x)}{h}\right]} + \underset{\substack{\downarrow \\ u(x)}}{u(x)} \underset{\substack{\downarrow \\ v'(x)}}{\left[\frac{v(x+h) - v(x)}{h}\right]}$$

$$= v(x)\,u'(x) + u(x)\,v'(x)$$

∎

ie: $$\boxed{\text{Prod Rule:} \quad \begin{aligned} &y = uv \\ &\frac{dy}{dx} = uv' + vu' \end{aligned}}$$

↑

Proof so you can see where this comes from

Lets check this with something we know

eg: $y = x^5$ (we know $\frac{dy}{dx} = 5x^4$)

Think of $y = x^5$ as a product.

$y = x^2 . x^3$

$\therefore$ Let $u = x^2$ $\quad v = x^3$

$u' = 2x$ $\quad v' = 3x^2$

$$\therefore \frac{dy}{dx} = uv' + vu'$$

$$= x^2(3x^2) + x^3(2x)$$

$$= 3x^4 + 2x^4$$

$$= 5x^4$$

as expected.

Note:

$$y = \underset{\substack{\uparrow \\ \text{deriv is} \\ 2x}}{x^2} . \underset{\substack{\uparrow \\ \text{deriv is} \\ 3x^2}}{x^3}$$

$$\xcancel{\frac{dy}{dx} = 2x \times 3x^2}$$

Do NOT multiply derivatives

Instead use the product rule.

The product rule tells us how to differentiate two functions that are multiplied.

eg: $y = (x^3+2x)(x^4-7)$

using product rule: Let $u = x^3+2x$ $\quad v = x^4-7$

$u' = 3x^2+2$ $\quad v' = 4x^3$

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu' \\ &= (x^3+2x)(4x^3) + (x^4-7)(3x^2+2) \\ &= 4x^6 + 8x^4 + 3x^6 - 21x^2 + 2x^4 - 14 \\ &= 7x^6 + 10x^4 - 21x^2 - 14\end{aligned}$$

Notice: If I expanded first $y = (x^3+2x)(x^4-7)$

$= x^7 - 7x^3 + 2x^5 - 14x$

+ then differentiated:

$$\frac{dy}{dx} = 7x^6 - 21x^2 + 10x^4 - 14$$

I get the same thing.

eg 2) $g(x) = (x^3 - x^5)(3x + 1)$

Let $u = x^3 - x^5$ $\quad v = 3x + 1$

$u' = 3x^2 - 5x^4$ $\quad v' = 3$

$$\frac{dg}{dx} = uv' + vu'$$

$$= (x^3 - x^5)(3) + (3x+1)(3x^2 - 5x^4)$$

$$= 3x^3 - 3x^5 + 9x^3 - 15x^5 + 3x^2 - 5x^4$$

$$= -18x^5 - 5x^4 + 12x^3 + 3x^2$$

3) $h(x) = \sqrt{x}\,(4x^2 - 5)$

Let $u = \sqrt{x} = x^{1/2}$ $\quad v = 4x^2 - 5$

$u' = \frac{1}{2}x^{-1/2}$ $\quad v' = 8x$

$$\frac{dh}{dx} = uv' + vu'$$

$$= x^{1/2}(8x) + (4x^2 - 5)\frac{1}{2}x^{-1/2}$$

$$= 8x^{3/2} + 2x^{3/2} - \frac{5}{2}x^{-1/2}$$

$$= 10x^{3/2} - \frac{5}{2}x^{-1/2}$$

Try Q1

Quotient Rule

If $y = \dfrac{u(x)}{v(x)}$

then $\dfrac{dy}{dx} = \dfrac{v(x)\,u'(x) - u(x)\,v'(x)}{v^2(x)}$

The proof uses a similar technique to the product rule proof (Try it yourself as an exercise!)

ie: $y = \dfrac{u}{v}$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} \longleftarrow \text{The order matters.}$$

eg: $y = \dfrac{x^2}{3x+1}\begin{array}{l}\longleftarrow u\\ \longleftarrow v\end{array}$

$u = x^2$ $\quad v = 3x+1$

$u' = 2x$ $\quad v' = 3$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{(3x+1)(2x) - x^2(3)}{(3x+1)^2}$$

$$= \frac{6x^2 + 2x - 3x^2}{(3x+1)^2}$$

$$= \frac{3x^2 + 2x}{(3x+1)^2}$$

eg 2) $y = \dfrac{x^4 + 3}{2x - 7}$

$u = x^4 + 3 \qquad v = 2x + 7$

$u' = 4x^3 \qquad v' = 2$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{\overset{\swarrow v}{(2x-7)}\overset{\swarrow u'}{(4x^3)} - \overset{\swarrow u}{(x^4+3)}\overset{\swarrow v'}{(2)}}{(2x-7)^2}$$

$$= \frac{8x^4 - 28x^3 - 2x^4 - 6}{(2x-7)^2}$$

$$= \frac{6x^4 - 28x^3 - 6}{(2x-7)^2}$$

3) $y = \dfrac{\sqrt{x}}{4x^2+3}$

$u = \sqrt{x} \qquad v = 4x^2 + 3$

$u' = \frac{1}{2}x^{-1/2} \qquad v' = 8x$

$$\frac{dy}{dx} = \frac{(4x^2+3)\frac{1}{2}x^{-1/2} - x^{1/2} \cdot 8x}{(4x^2+3)^2}$$

$$= \frac{\frac{1}{2}x^{-1/2}(4x^2+3) - 8x^{3/2}}{(4x^2+3)^2}$$

Try Q2

Chain Rule

- We've seen how to differentiate products and quotients.
- What about $y = (x^2+1)^5$

Think of this as a function inside a function.

ie! x^2+1 inside x^5 function

(Remember composition of functions)

- There is another rule which we use to differentiate these types of functions.

Chain Rule (or Function of a function Rule)

If $y = f(u)$ and $u = g(x)$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

eg) $y = (x^2+1)^5$

Let $u = x^2+1 \longrightarrow \dfrac{du}{dx} = 2x$

so $y = u^5 \longrightarrow \dfrac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times 2x = 5(x^2+1)^4 \cdot 2x$$

$$= 10x(x^2+1)^4$$