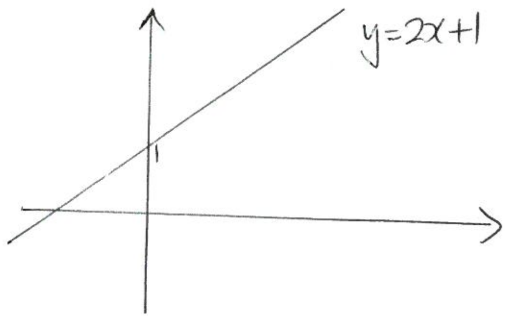
Intro to Differentiation

Recall Lines :



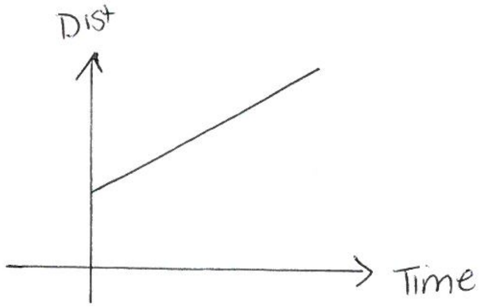
gradient = $\frac{\text{rise}}{\text{run}}$

$= \frac{\text{change in } y}{\text{change in } x}$

= rate of chang of y wrt x

gradient - useful to help us understand how things are changing

Eg :

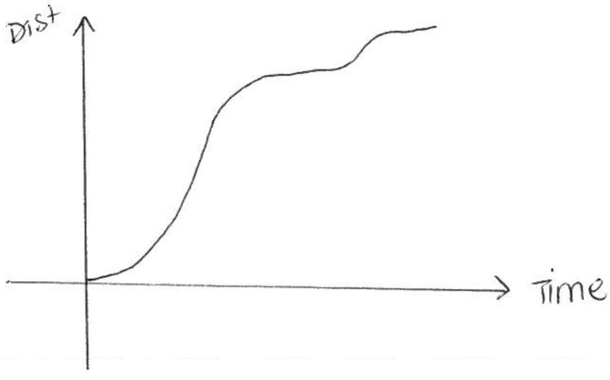


here gradient = speed

$= \frac{\text{change in dist}}{\text{Change in time}}$

Eg : Policeman + Motorist story

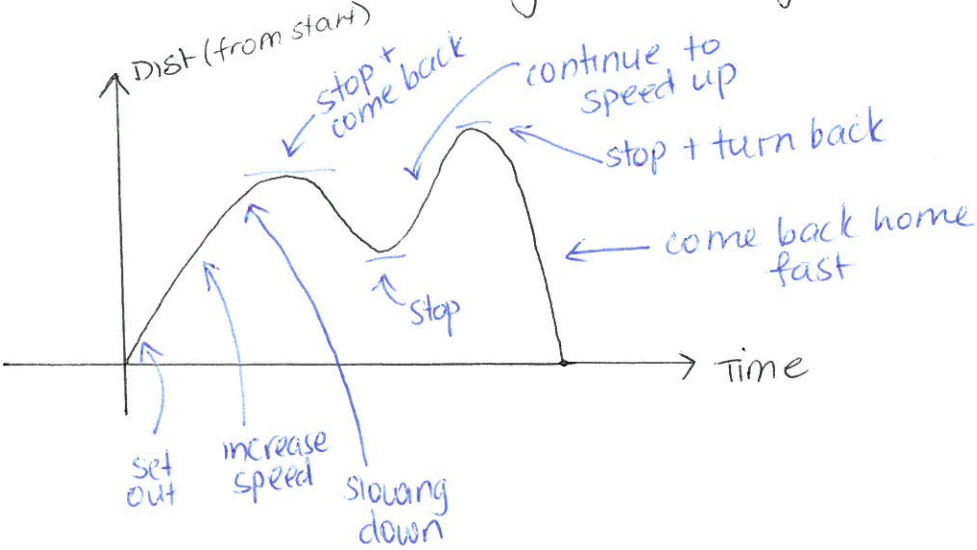
more realistic picture of drive



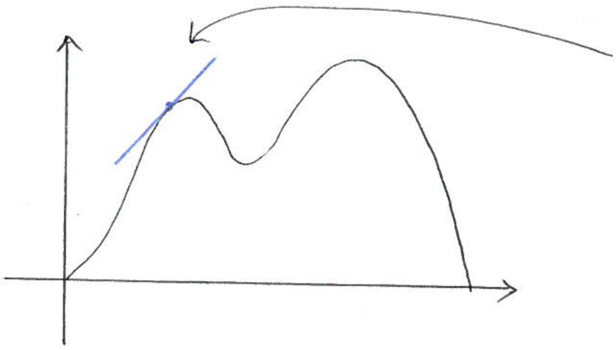
- The speed we talked about earlier is the average speed.
- It would be more useful to calculate the speed at a particular instant.

When we travel it is not usually a straight line.

More like:



- Depending on where we are, our speed is different
- If we want the instantaneous speed then we are more interested in the slope at a particular point on the curve



slope at this point

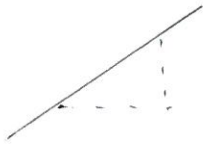
=

slope of line just touching curve at this point

=

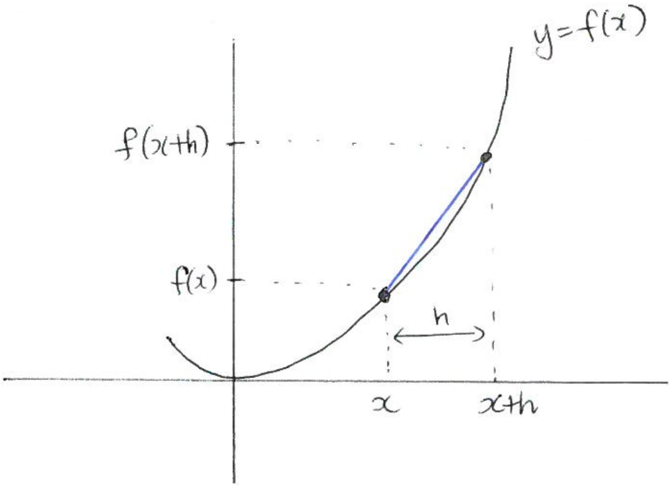
this line is called a tangent

Mathematical construction:

start with what we know : Lines  slope $= \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$

$\frac{\text{change in } y}{\text{change in } x}$

We want the slope of a curve



start by approximating it:

$$\text{slope} = \frac{f(x+h) - f(x)}{h}$$

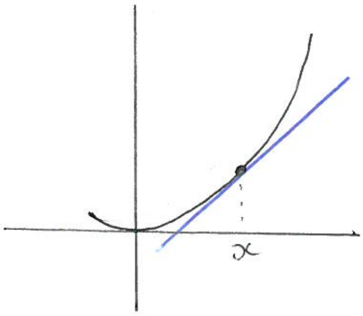
To get a better approximation we want to move the points closer together

ie: distance between them gets less

ie: h gets closer to 0

ie: $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Limit - we're saying let $h \to 0$



when $h \to 0$ we get the slope of the tangent at this point

This slope is known as our derivative

we say $$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$f'(x)$ = notation for our derivative

Also use y' or $\frac{dy}{dx}$ ← $\frac{\text{change in } y}{\text{change in } x}$ at an instant

Derivative = rate of change

= slope of tangent to curve

= gradient at each point x of function

Differentiation = process of finding the derivative

We say a function is Differentiable if this limit exists.

eg: $f(x) = x^2$

So $f(x+h) = (x+h)^2$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

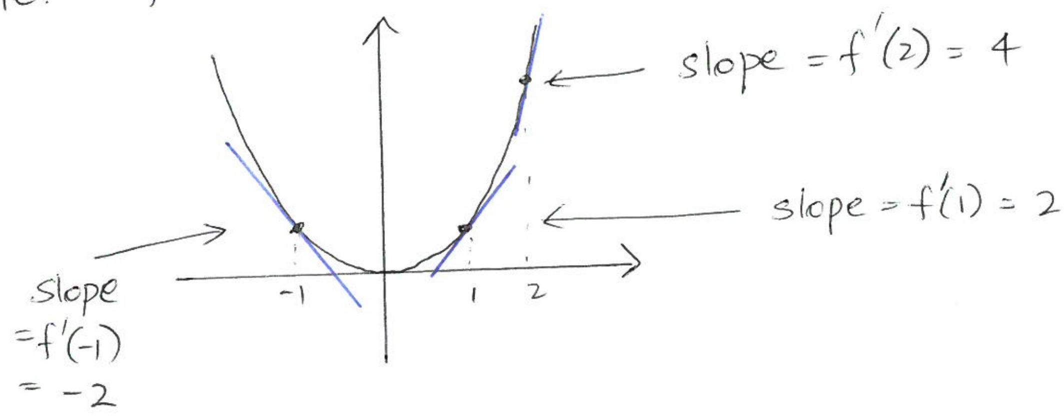
$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 2x$$

$\therefore f'(x) = 2x$.

- This process we just went through is called differentiation
- So $f'(x) = 2x$ gives us a formula for the gradient at each point of our function $f(x) = x^2$

ie: $f(x) = x^2$



Using $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is called

"Differentiation by First principles" or differentiation using the "Definition of the Derivative."

Eg1 a) $f(x) = x^2 + 3x + 2$

$f(x+h) = (x+h)^2 + 3(x+h) + 2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 2 - (x^2+3x+2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h \cancel{+2} \cancel{-x^2} \cancel{-3x} \cancel{-2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 3)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h + 3$$

$$= 2x + 3$$

$$\therefore f'(x) = 2x + 3$$

b) $g(x) = x^3$

$\therefore g(x+h) = (x+h)^3$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= 3x^2$$

c) $f(x) = \frac{1}{x}$

$\therefore f(x+h) = \frac{1}{x+h}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\left[\frac{x - (x+h)}{x(x+h)}\right]}{h}$$

$$= \lim_{h \to 0} \frac{-h}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)}$$

$$= \frac{-1}{x^2}$$

d) $f(x) = 4x^2 + 1$

$\therefore f(x+h) = 4(x+h)^2 + 1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{4(x+h)^2 + 1 - (4x^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{4(x^2 + 2xh + h^2) + 1 - 4x^2 - 1}{h}$$

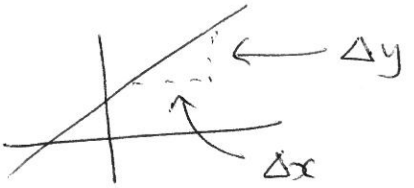
$$= \lim_{h \to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 + \cancel{1} - \cancel{4x^2} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(8x + 4h)}{\cancel{h}}$$

$$= 8x$$

Notation:

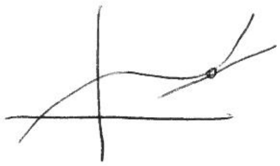


slope $= \dfrac{\Delta y}{\Delta x}$

this is the average rate of change.

$\dfrac{dy}{dx}$ = rate of change at an instant

ie: $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$



we say "dy on dx" (Not division)

$\dfrac{dy}{dx}$ tells us our function y is written in terms of x
and $\dfrac{dy}{dx}$ is the derivative of y with respect to x

Also equivalent are $\dfrac{df}{dx}$ or $\dfrac{d\,f(x)}{dx}$ or $\dfrac{d}{dx}(f(x))$

eg: we saw $f(x) = 4x^2+1$
+ $f'(x) = 8x$

I can also say $\dfrac{df}{dx} = 8x$

If $f(t) = 4t^2+1$ then $\dfrac{df}{dt} = 8t$