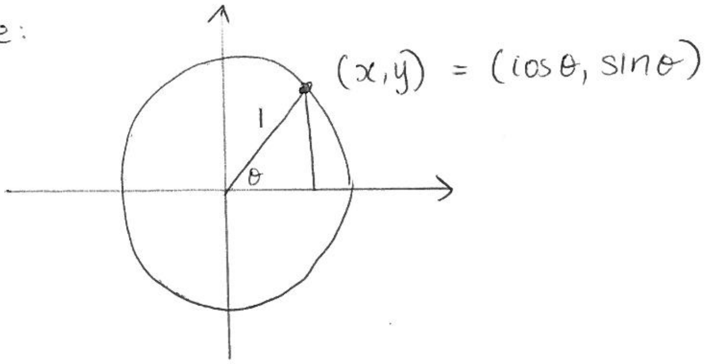
# Trig Identities

unit circle:




Notice by Pythagoras : $x^2+y^2=1$

ie: $\boxed{\cos^2\theta + \sin^2\theta = 1}$

one of the most common and well used identities in Trig.

Dividing both sides by $\cos^2\theta$ we get

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \rightarrow \text{ie: } \boxed{1 + \tan^2\theta = \sec^2\theta}$$

Dividing both sides by $\sin^2\theta$ we get

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta} \rightarrow \text{ie: } \boxed{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

These are the Pythagorean Identities:

$$\boxed{\begin{aligned} &\cos^2\theta + \sin^2\theta = 1 \\ &1 + \tan^2\theta = \sec^2\theta \\ &\cot^2\theta + 1 = \operatorname{cosec}^2\theta \end{aligned}}$$

# Addition Formulas

$$\boxed{\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B\end{aligned}}$$

See Proof in MUMS module 4.6!

Eg Find the exact value of

a) $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

b) $\cos \frac{7\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}}$$

c) $\sin \frac{\pi}{12} = \left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Eg Prove the identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= 0 \cdot \cos x + 1 \cdot \sin x$$
$$= \sin x$$

## More Identities

If we let $A = B$ in the addition formulas we get the double angle formulas.

ie: $\sin(A+A) = \sin A\cos A + \cos A\sin A \rightarrow \boxed{\sin 2A = 2\sin A\cos A}$

$\cos(A+A) = \cos A\cos A - \sin A\sin A \rightarrow \boxed{\cos 2A = \cos^2 A - \sin^2 A}$

Double Angle Formulas:

$$\boxed{\begin{aligned} \sin 2A &= 2\sin A\cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A \\ &= 2\cos^2 A - 1 \end{aligned}}$$

← we get these other versions by substituting the Pythagorean Id in eg) Find the exact value of $\cos \frac{2\pi}{3}$ using the double angle formula.

$$\cos \frac{2\pi}{3} = \cos 2\left(\frac{\pi}{3}\right) \qquad [\cos 2\theta = \cos^2\theta - \sin^2\theta)$$

this is $\theta$

$$= \cos^2 \frac{\pi}{3} - \sin^2 \frac{\pi}{3}$$

$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

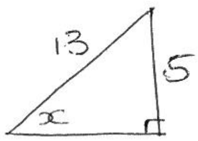
$$= \frac{1}{4} - \frac{3}{4}$$

$$= \frac{-2}{4}$$

$$= -\tfrac{1}{2}$$

eg $\sin x = \frac{5}{13}$ $\qquad \frac{\pi}{2} \leq x \leq \pi$

$x$ in 2nd quad. $\rightarrow$ sin pos, cos neg




$\sqrt{13^2 - 5^2} = 12$

$\therefore \cos x = \frac{-12}{13}$

$$\therefore \sin 2x = 2 \sin x \cos x$$

$$= 2\left(\frac{5}{13}\right)\left(\frac{-12}{13}\right)$$

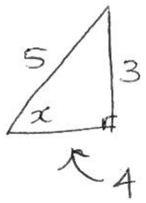
$$= -\frac{120}{169}$$

eg) Suppose $\sin x = \frac{3}{5}$ + $x$ is in the first quad.
Find $\sin 2x$, $\cos 2x$ and $\tan 2x$.

want $\sin 2x = 2 \sin x \cos x$

we know $\sin x = \frac{3}{5}$

ie:




$\therefore \cos x = \frac{4}{5}$

since $x$ is in 1st quad
they are positive

$$\therefore \sin 2x = 2 \sin x \cos x$$
$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$$
$$= \frac{24}{25}$$

and
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$$
$$= \frac{7}{25}$$

$$\therefore \tan 2x = \frac{\sin 2x}{\cos 2x}$$
$$= \frac{24/25}{7/25}$$
$$= \frac{24}{7}$$

Eg) Simplify

a) $\dfrac{\sin x \sec x}{\tan x} = \dfrac{\sin x \dfrac{1}{\cos x}}{\tan x}$

$= \dfrac{\tan x}{\tan x}$

$= 1$

b) $\dfrac{\sec x - \cos x}{\tan x} = \dfrac{\dfrac{1}{\cos x} - \cos x}{\dfrac{\sin x}{\cos x}}$

$= \dfrac{1 - \cos^2 x}{\cos x} \div \dfrac{\sin x}{\cos x}$

$= \dfrac{\sin^2 x}{\cancel{\cos x}} \times \dfrac{\cancel{\cos x}}{\cancel{\sin x}}$

$= \sin x$

Eg a) verify $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\operatorname{cosec} x} = 1$

LHS $= \dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\operatorname{cosec} x}$

$= \dfrac{\cos x}{1/\cos x} + \dfrac{\sin x}{1/\sin x}$

$= \cos^2 x + \sin^2 x$

$= 1$

$=$ RHS.

b) Verify $\tan 2\theta = \frac{2\sin\theta\cos\theta}{\cos 2\theta}$

$$RHS = \frac{2\sin\theta\cos\theta}{\cos 2\theta}$$

$$= \frac{\sin 2\theta}{\cos 2\theta} \quad \text{(double angle)}$$

$$= \tan 2\theta$$

$$= LHS$$

c) Verify $\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

$$LHS = \frac{\cos\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cancel{\cos\theta}(1+\sin\theta)}{\cos^{\cancel{2}}\theta}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta$$