Sum of a Geometric Progression

Sum of the first n terms of a GP is

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Proof : Take a GP $a, ar, ar^2, \cdots, ar^{n-1}, \cdots$

So $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ — ①

Now $rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$ — ②

① − ② : $S_n - rS_n = a - ar^n$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

(② − ① gives the second formula)

Eg a) Find the sum of first 10 terms of $3, 6, 12, 24, \cdots$

This is a GP with $a = 3, r = 2$

Want $S_{10} = \frac{a(r^{10}-1)}{r-1}$

$$= \frac{3(2^{10}-1)}{2-1}$$

$$= 3(2^{10}-1)$$

b) Find sum of first 8 terms of $a_n = 2^n$

so $a_n = 2^n$ is the seq : $2, 4, 8, 16, \ldots$

ie: $a=2$ $r=2$

$$\therefore S_8 = \frac{2(2^8-1)}{2-1}$$

$$= 2(2^8-1)$$

c) $\frac{8}{9} + \frac{4}{3} + 2 + \ldots$ Find S_6.

↑
This is a GP with $a = \frac{8}{9}$ $r = \frac{4/3}{8/9} = \frac{4}{3} \div \frac{8}{9}$

$$= \frac{4}{3} \times \frac{9}{8}$$

$$= \frac{3}{2}$$

check $\frac{2}{4/3} = 2 \div \frac{4}{3} = 2 \times \frac{3}{4} = \frac{3}{2}$

$$\therefore S_6 = \frac{a(r^6-1)}{r-1}$$

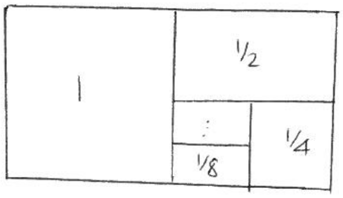
$$= \frac{8/9\left((3/2)^6-1\right)}{3/2-1}$$

$$= \frac{8/9\left((3/2)^6-1\right)}{1/2}$$

$$= \frac{16}{9}\left(\left(\frac{3}{2}\right)^6-1\right)$$

Limiting Sum

- An interesting thing happens when $|r| < 1$



$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \rightarrow 2$

↑ approaches

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ actually approaches 2

- Look at this algebraically

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

If $|r| < 1$ then r^n gets smaller + smaller.

eg: $r = \frac{1}{2}$ then r^n : $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots \rightarrow 0$

So when we sum an infinite number of terms $r^n \rightarrow 0$

$\therefore \quad S_n \rightarrow \dfrac{a}{1 - r}$ ← called the limiting sum

$$\boxed{\text{when } -1 < r < 1 \text{ the limiting sum exists and } S_\infty = \frac{a}{1-r}}$$

↖ Limiting Sum only relevant for GPs

Eg) Find the limiting sums

a) $12+6+3+\cdots$

$a=12$; $r=\frac{6}{12}=\frac{3}{6}=\frac{1}{2}$

Since $-1<r<1$ we can find the limiting sum

$$S_\infty=\frac{a}{1-r}=\frac{12}{1-1/2}=\frac{12}{1/2}=24$$

b) $200-40+8-\cdots$

$a=200$ $r=\frac{-40}{200}=\frac{8}{-40}=-\frac{1}{5}$

Since $-1<r<1$ we can find $S_\infty=\frac{a}{1-r}$

$$=\frac{200}{1-(-1/5)}$$

$$=\frac{200}{6/5}$$

$$=\frac{1000}{6}$$

$$=\frac{500}{3}$$

Summary:

	Arithmetic	Geometric
	adds constant	multiplies by const
test	$a_2-a_1=a_3-a_2=\cdots=a_n-a_{n-1}$ $=d$	$\frac{a_2}{a_1}=\frac{a_3}{a_2}=\cdots=\frac{a_n}{a_{n-1}}=r$
nth term	$a_n=a+(n-1)d$	$a_n=ar^{n-1}$
sum of n terms	$S_n=\frac{n}{2}(2a+(n-1)d)$ $S_n=\frac{n}{2}(a+\ell)$	$S_n=\frac{a(1-r^n)}{1-r}$
		Limiting sum $S_\infty=\frac{a}{1-r}$ if $-1<r<1$

Sigma Notation

= a shorthand notation for taking the sum

Σ ← indicates sum

eg: $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

↑ expression for terms in sum

boundaries - start at $k=1$ + go up to $k=5$

eg: $\sum_{k=1}^{10} 3k = 3 + 6 + 9 + \cdots + 30$

same as $\sum_{i=1}^{10} 3i = 3 + 6 + 9 + \cdots + 30$

We can find $\sum_{i=1}^{n} 3i = 3 + 6 + 9 + \cdots$

↑ sum of first n terms

this is an AP : $a = 3$, $d = 3$

so $S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}(2(3) + (n-1)3)$

$= \frac{n}{2}(6 + 3n - 3)$

$= \frac{n}{2}(3 + 3n)$

Eg

b) $\sum_{m=1}^{25} 2m+3 = \underset{m=1}{5} + \underset{m=2}{7} + \underset{m=3}{9} + \cdots + 53$

this is an AP with $a=5$, $d=2$, $l=53$

$$\therefore S_{25} = \frac{25}{2}(5+53)$$

$$= \frac{25}{2}(58)$$

$$= 725$$

c) $\sum_{k=1}^{6} 2 \cdot 3^{k-1} = 2 \cdot 3^0 + 2 \cdot 3^1 + 2 \cdot 3^2 + \cdots$

$= 2 + 6 + 18 + \cdots$

$=$ GP with $a=2$, $r=3$, $n=6$

$$S_6 = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{2(3^6 - 1)}{3-1}$$

$$= 3^6 - 1$$

d) $\sum_{k=1}^{\infty} 2^{-k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$=$ GP with $a = 1/2$, $r = 1/2$

$S_\infty = \frac{a}{1-r}$ which exists since $|r|<1$

$$\therefore S_\infty = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

Eg: Ann commences work on a salary of \$31200. Each year her salary increases by \$1200.

a) Find her salary in the tenth year

b) Find the total amount shei would have earned after 10 yrs.

31200 , 32400 , 33600 , ...

1st yr, 2nd yr, 3rd yr

AP with $a = 31200$, $d = 1200$

a) Salary in 10 yr → want a_{10} – tenth term

$$\begin{aligned} a_{10} &= a + (10-1)d \\ &= 31200 + 9(1200) \\ &= 42\,000 \end{aligned}$$

b) Total after 10 yrs = S_{10} → sum of is ten terms

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\begin{aligned} S_{10} &= \frac{10}{2}\left(2(31200) + (10-1)1200\right) \\ &= 5(62400 + 10800) \\ &= 366\,000 \end{aligned}$$

Eg A leak develops in a dam. On the first day 120 L of water escape. Each day thereafter the amount that escapes is 80% of the amount of the previous day.

a) How much water will have escaped after 20 days?

b) How much water will escape altogether?

Day 1 : 120 L

Day 2 : $120 \times 0.8 = 96$

Day 3 : $96 \times 0.8 = 76.8$

$\vdots$

ie: $120 \underset{\times 0.8}{\curvearrowright} 96 \underset{\times 0.8}{\curvearrowright} 76.8, \ldots$ GP with $a = 120$, $r = 0.8$

a) Water escaped after 20 days $= S_{20}$

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{so} \quad S_{20} = \frac{120(1-0.8^{20})}{1-0.8}$$

$$= \frac{120(1-0.8^{20})}{0.2}$$

$$= 593.08 \text{ L}$$

b) Total water escaped = sum to infinity

this exists since $-1 < r < 1$

$$\therefore S_\infty = \frac{a}{1-r} = \frac{120}{1-0.8}$$

$$= 600 \text{ L}$$