Polynomials continued

We saw : If we knew 1 root

→ we can get 1 factor

→ we can get other factors

→ we can factorise poly

eg) Factorise $p(x) = x^3 + x^2 - 14x - 24$ given $x = 4$ is a root.

note: we can confirm $x = 4$ is a root by looking at $p(4) = 0$

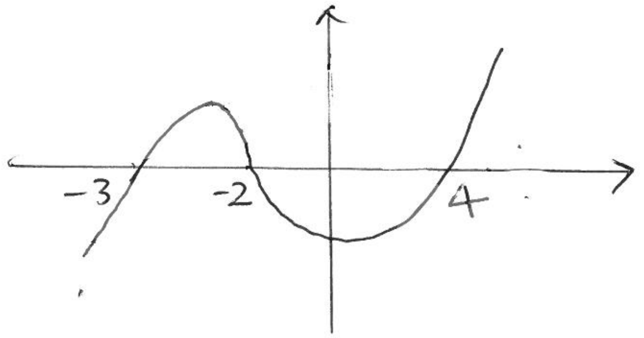
$\therefore x = 4$ is a root $\Rightarrow$ $x - 4$ is a factor

$$
\begin{array}{r|l}
 & x^2 + 5x + 6 \\
\hline
x - 4 & x^3 + x^2 - 14x - 24 \\
 & x^3 - 4x^2 \\
\hline
 & 5x^2 - 14x - 24 \\
 & 5x^2 - 20x \\
\hline
 & 6x - 24 \\
 & 6x - 24 \\
\hline
 & 0
\end{array}
$$

0 ← we expect this.

So $p(x) = (x-4)(x^2+5x+6)$

$= (x-4)(x+2)(x+3)$



But how do we factorise a poly if we don't know 1 root.

Notice: $(2x+3)(5x+4) = 10x^2 + 23x + 12$

make up leading term

make up constant term

Solutions $= -\frac{3}{2}, -\frac{4}{5}$ ← factors of constant term
← factors of leading coeff

Similarly $(3x+1)(x-4)(2x+5) = 6x^3 + \cdots -20$

$3 \times 1 \times 2$ $\quad$ $1 \times 4 \times 5$

Solutions $= -\frac{1}{3}, \frac{4}{1}, \frac{-5}{2}$ ← factors of const term
← factors of leading coeff

Theorem: If $p(x) = a_n x^n + \cdots + a_0$ where $a_0, a_1, \cdots, a_n$ are integer coefficients,

Then every rational root is of the form $\frac{m}{n}$ where $m =$ factors of a_0
$n =$ factors of a_n

Finding Solutions

When using a theorem we have to make sure we satisfy the conditions.

* poly must have integer coefficients
* this gives us rational roots.

∴ use this theorem to find the initial solution.

Look at possibilities : $\pm \dfrac{\text{factors of constant term}}{\text{factors of leading coeff.}}$

+ then use trial + error to find a root from these possibilities eg13) Solve $2x^3 - 3x^2 - 23x + 12 = 0$.

First find one soln

$$\text{Possibilities} = \pm \frac{\text{factors of } 12}{\text{factors of } 2} \begin{array}{l} \leftarrow 1, 2, 6, 3, 4, 12 \\ \leftarrow 1, 2 \end{array}$$

$\therefore$ Possibilities are $\pm 1, 2, 6, 3, 4, 12, \frac{1}{2}, \frac{3}{2}$

Try $x = 1$: $2(1)^3 - 3(1)^2 - 23(1) + 12$

$= 2 - 3 - 23 + 12$

$= 12$

$\neq 0$ $\therefore$ 1 is not a soln.

$x = -3$: $2(-3)^3 - 3(-3)^2 - 23(-3) + 12$

$= -2(27) - 3(9) + 69 + 12$

$= -54 - 27 + 69 + 12$

$= 0$

✓

$\therefore$ $x = -3$ is a soln

$\therefore$ $x - (-3)$ is a factor

$x + 3$ " " "

Finding other solutions:

$$
\begin{array}{r|l}
 & 2x^2 - 9x + 4 \\
\hline
x+3 & 2x^3 - 3x^2 - 23x + 12 \\
 & \underline{2x^3 + 6x^2} \\
 & \quad -9x^2 - 23x + 12 \\
 & \quad \underline{-9x^2 - 27x} \\
 & \qquad 4x + 12 \\
 & \qquad \underline{4x + 12} \\
 & \qquad\qquad 0
\end{array}
$$

$\therefore$ $(x+3)(2x^2-9x+4) = 0$

$(x+3)(2x-1)(x-4) = 0$

$\therefore x = -3, \frac{1}{2}, 4$

Note: Instead of long division.

$$2x^3 - 3x^2 - 23x + 12 = (x+3)(\underset{\substack{\uparrow \\ \text{has to be } 2}}{\_}x^2 + \_x + \underset{\substack{\uparrow \\ \text{has to be } 4}}{\_})$$

$$= (x+3)(2x^2 + ax + 4)$$

$\therefore$ Expand to find a: $2x^3 + ax^2 + 4x + 6x^2 + 3ax + 12$

$= 2x^3 + (6+a)x^2 + (3a+4)x + 12$

$\therefore 6+a = -3 \rightarrow a = -9$

$3a+4 = -23 \rightarrow a = -9$

$\therefore$ $(x+3)(2x^2-9x+4)$

Eg14) Consider $p(x) = 18x^3 + 3x^2 - 4x - 1$

a) Find 1 root using trial + error
b) Hence factorise this polynomial
c) Find all solutions of $p(x) = 0$
d) Sketch the graph of $p(x)$.

a) Possible roots $= \pm \dfrac{\text{factors of } 1}{\text{factors of } 18}$ ← 1, 2, 9, 3, 6, 18

$= \pm 1, \frac{1}{2}, \frac{1}{9}, \frac{1}{3}, \frac{1}{6}, \frac{1}{18}$

Try $x = \frac{1}{2}$: $p(\frac{1}{2}) = 18(\frac{1}{2})^3 + 3(\frac{1}{2})^2 - 4(\frac{1}{2}) - 1$

$= \frac{18}{8} + \frac{3}{4} - 2 - 1$

$= 0$

$\therefore x = \frac{1}{2}$ is a root.

b) $\therefore x - \frac{1}{2} = 0$

ie: $2x - 1 = 0$

$\therefore$ Take $2x - 1$ as a factor of $p(x)$

$$\begin{array}{r|l} & 9x^2 + 6x + 1 \\ \hline 2x-1 & 18x^3 + 3x^2 - 4x - 1 \\ & \underline{18x^3 - 9x^2} \\ & \quad 12x^2 - 4x - 1 \\ & \quad \underline{12x^2 - 6x} \\ & \qquad\quad 2x - 1 \\ & \qquad\quad \underline{2x - 1} \\ & \qquad\qquad 0 \end{array}$$

So $p(x) = (2x-1)(9x^2+6x+1)$

Notice perf square

$(3x)^2 + 2(3x)(1) + 1^2$

$= (3x+1)^2$

$\therefore p(x) = (2x-1)(3x+1)^2$

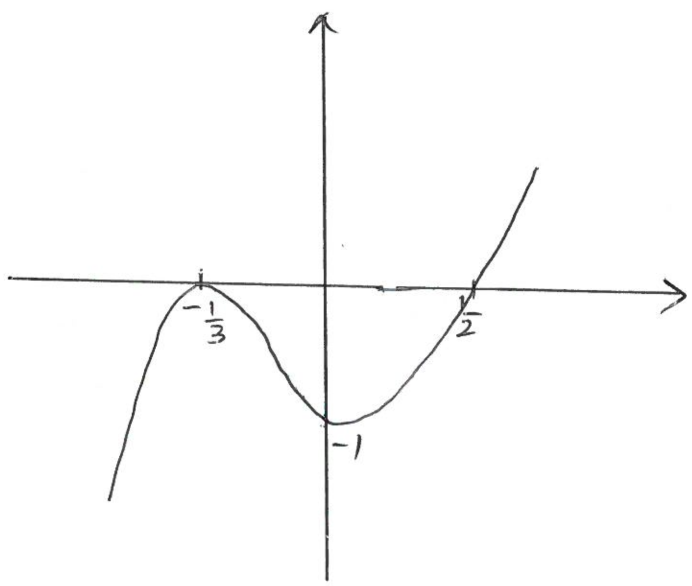
c) Solutions $(2x-1)(3x+1)^2 = 0$

$\therefore$ $2x-1 = 0$ $\quad$ $3x+1 = 0$

$x = 1/2$ $\quad$ $x = -1/3$

(2 solutions, $x = -1/3$ is a double root)

d)



(test: $x=0$: $p(x) = -1$