Polynomials continued

Eg. Divide $p(x) = x^3 + x^2 + x + 5$ by $f(x) = x - 1$

$$
\begin{array}{r l}
 & x^2 + 2x + 3 \\
x - 1 \,) & x^3 + x^2 + x + 5 \\
 & \underline{x^3 - x^2} \\
 & \quad 2x^2 + x + 5 \\
 & \quad \underline{2x^2 - 2x} \\
 & \qquad 3x + 5 \\
 & \qquad \underline{3x - 3} \\
 & \qquad\quad 8
\end{array}
$$

$q(x) = x^2 + 2x + 3$

$r(x) = 8$

So we can say $p(x) = (x-1)\,q(x) + r(x)$

where $q(x) = x^2 + 2x + 3$

$r(x) = 8$

Notice :

- deg $r(x)$ must be smaller than divisor
  - Here we expect $r(x) =$ constant
  - ie: we can say $p(x) = (x-1)\,q(x) + r$

- Notice we divided by $x - 1$

  $\therefore$ Lets let $x = 1$: $p(1) = (1-1)\,q(1) + r$

  $= r$

  $= 8$

  $\therefore$ when dividing by $x - 1$ the remainder is $p(1)$

Remainder Theorem

Th: If a polynomial $p(x)$ is divided by $x-a$, then the remainder is $p(a)$

Proof: when $p(x)$ is divided by $x-a$ we get

$$p(x) = (x-a)\,q(x) + r$$

Let $x=a$: $p(a) = (a-a)q(a) + r$

$$= r$$

ie: The remainder $= p(a)$

polynomial evaluated at a

note: When using this theorem we must make sure we use it for the appropriate factor.

- make sure divisor is of the form $x-a$
- cant use it when divisor is x^2-1 or $2x+5$ etc.

eg8) Find the remainder when $p(x) = 3x^3+5x^2-x+4$ is divided by $x+2$.

Here divisor is $x+2 = x-(-2)$

$\qquad\qquad\qquad\qquad\quad\; a$

$\therefore$ Rem $= p(-2) = 3(-2)^3 + 5(-2)^2 - (-2) + 4$

$= -24 + 20 + 2 + 4$

$= 2$

(Note: This confirms what we got when we used long div)

9) Find the remainder when $p(x) = 2x^4 - x^3 + 1$ is divided by $x+3$.

Using Remainder Theorem, divisor is $x+3 = x-(-3)$

$\therefore$ Rem $= p(-3) = 2(-3)^4 - (-3)^3 + 1$

$= 2(81) - (-27) + 1$

$= 190$

10) Use the Remainder Theorem to show that $x+2$ is a factor of $p(x) = x^5 + 2x + 36$.

$\therefore \quad p(-2) = (-2)^5 + 2(-2) + 36$

$= -32 - 4 + 36$

$= 0$

since $x+2$ is a factor there is no remainder

So $p(-2) = 0$ + there is no remainder

$\therefore$ $x-(-2)$ fully divides into $p(x)$

$\therefore$ $x+2$ is a factor of $p(x)$

Factor Theorem

$\boxed{\text{Suppose } p(x) \text{ is a poly with } p(a)=0. \text{ Then } x-a \text{ is a factor of } p(x).}$

Proof: Suppose $p(a)=0$

Then by Rem Th: $p(x) = (x-a)\,q(x) + 0$

$= (x-a)\,q(x)$

$\therefore\ x-a$ is a factor of $p(x)$

eg11) Show that $x-3$ is a factor of $p(x) = 3x^4 - 8x^3 - 14x^2 + 31x + 6$

(If $x-3$ is a factor we expect $p(3)=0$)

$\therefore\ p(3) = 3(3)^4 - 8(3)^3 - 14(3)^2 + 31(3) + 6$

$= 243 - 216 - 126 + 93 + 6$

$= 0.$

Notice: we have actually found a solution to $p(x)=0$

since $p(3)=0$

$\therefore\ x=3$ is a solution

ie: $x=3$ is a root of $p(x)$

ie: $x=3$ is a zero of $p(x)$

+ $x-3$ is a factor of $p(x)$

ie: $\boxed{\text{If } x=a \text{ is a root, then } x-a \text{ is a factor.}}$

eg 2) Show that $x=3$ is a root of the poly $p(x)=x^3-x^2-41x+105$ and find all other roots.

· Using the factor theorem we see $p(3)=3^3-3^2-41(3)+105$

$$=27-9-123+105$$

$$=0$$

$\therefore$ $x=3$ is a root of $p(x)$

+ $x-3$ " " factor " "

· use long division to find other factors

$$
\begin{array}{r}
x^2+2x-35 \\
x-3\ \big)\ \overline{x^3-x^2-41x+105} \\
\underline{x^3-3x^2} \qquad\qquad \\
2x^2-41x \qquad \\
\underline{2x^2-6x} \qquad \\
-35x+105 \\
\underline{-35+105} \\
0
\end{array}
$$

$\therefore$ $p(x) = (x-3)(x^2+2x-35)$

· To find other zeros we want to solve

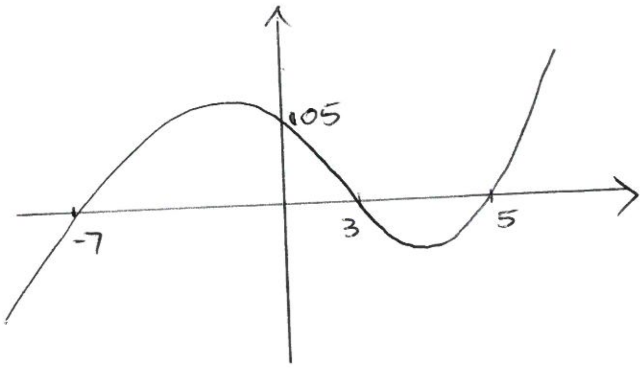
$$(x-3)(x^2+2x-35)=0$$

ie: $(x-3)(x+7)(x-5)=0$

$\therefore$ $x=3,-7,5$

Note: • We even know how to sketch this

- It's a cubic ~ or ~
- roots gives us x intercepts



(test point : $x=0$: $y = 105$)

• Notice • once we can find one root then we can use long division + get the other solutions

• Degree of poly tells us max n° of roots.

• what if we didn't know the initial root?

How can we guess a root?