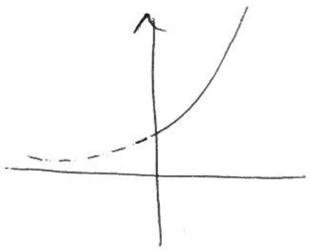
Applications

- many real life problems are exponential models.



exponential growth

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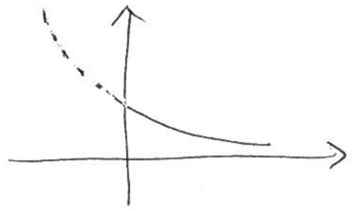
eg: - population growth
  → bacteria
  → virus
  → human population
  → pop of species
- economic growth
- compound interest



exponential decay

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eg: - radio active decay
- heat transfer
  → Newtons law of cooling
- atmospheric pressure (decreases exp with increasing height above sea level)

model : $y = Ae^{kt}$

k = growth constant

For $k>0 \rightarrow$ exp growth

$k<0 \rightarrow$ exp decay

8. $P(t) = 15600e^{0.09t}$ describes the population of a city t years after 1984.

(a) What will the population be in 1994?

(b) How long will it take for the population to reach 100,000?

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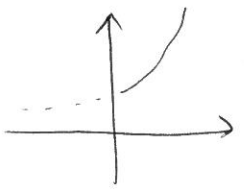
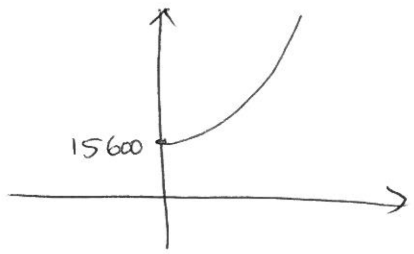
Make sure you understand the model.

- This is exponential growth since $k = 0.09$ is pos



- Initially → $t = 0$: $P = 15600\,e^0 = 15600$

So in 1984 $P = 15600$



When $t = 1$ (ie. 1985) $P = 15600\,e^{0.09(1)} = 17069$

$t = 2$ $P = 15600\,e^{0.09(2)} = 18676$ $\times e^{0.09}$

$t = 3$ $P = 15600\,e^{0.09(3)} = 20435$ $\times e^{0.09}$

⋮ ⋮

a) we want population in 1994

$\rightarrow$ ie: want P when $t = 10$.

$$P(t) = 15600\, e^{0.09t}$$

$$\therefore P(10) = 15600\, e^{0.09(10)}$$

$$= 38\,369 \quad \text{(nearest whole n}^{o}\text{)}$$

b) How long for population to reach 100 000 ?

ie: want t when $P = 100\,000$

ie: $100\,000 = 15600\, e^{0.09t}$

ie: $\dfrac{100\,000}{15600} = e^{0.09t}$

$$\ln\left(\frac{1000}{156}\right) = \ln e^{0.09t}$$

$$\therefore 0.09t = \ln\left(\frac{1000}{156}\right)$$

$$t = \frac{\ln\left(\frac{1000}{156}\right)}{0.09}$$

$$= 20.64$$

$\therefore$ It will take 20.6 years

9. The number of bacteria in a population, given by the formula $N(t) = N_0 e^{0.12t}$, has an initial population of 240000, where t is measured in hours. How long will it take for the population of bacteria to reach 250000?

↑
Remember this ques from wk 9.

$$N(t) = N_0 e^{0.12t}$$

N = number of bacteria
t = time (hours)

We saw when $t = 0$, $N = 240\,000$

So the initial value $N_0 = 240\,000$

$\therefore$ Our model looks like $N(t) = 240\,000\, e^{0.12t}$

We want to know how long it will take for $N = 250\,000$

ie: want t when $N = 250\,000$

$$\therefore \quad 250\,000 = 240\,000\, e^{0.12t}$$

$$\frac{250\,000}{240\,000} = e^{0.12t}$$

ie: $$e^{0.12t} = \frac{25}{24}$$

$$\ln e^{0.12t} = \ln\left(\frac{25}{24}\right)$$

$$0.12t = \ln\left(\frac{25}{24}\right)$$

$$t = \frac{\ln(25/24)}{0.12}$$

$$= 0.341 \text{ hrs}$$

ie: It will take 0.34 hr or $\approx$ 20 mins for the pop to reach 250000.

10. When a certain medical drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modelled by $D(t) = 50e^{-0.2t}$.

(a) How many milligrams in the initial dose?

(b) How may milligrams will remain in the bloodstream after 3 hours?

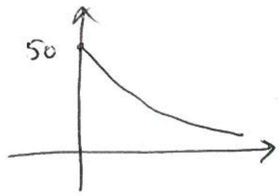
(c) The patient needs to take a second dose of the drug once there is less than 5mg in their bloodstream. How many hours later does the second dose need to be administered?

$D(t) = 50e^{-0.2t}$

D = drug dose (mg)
t = time (hours)

notice this is decay — form a picture of what is happening

a) initial dose → want D when $t=0$



$D(0) = 50e^{0} = 50$

∴ Initial dose was 50mg

b) After 3 hours → want D when $t=3$.

$D(3) = 50e^{-0.2(3)}$

$= 27.4$ mg

c) want time when less than 5mg

ie: Find t when $D = 5$

∴ $50e^{-0.2t} = 5$

$e^{-0.2t} = \frac{5}{50}$

$\ln e^{-0.2t} = \ln\left(\frac{1}{10}\right)$

$-0.2t = \ln(0.1)$

$t = \frac{\ln(0.1)}{-0.2}$

$= 11.51$

∴ Second dose should be administered after 11.51 hrs

1. The number of a certain species of frog is modelled by the function $N(t) = 85e^{0.18t}$ where t is measured in years.

(a) What is the initial population of frogs?

(b) What will the population be after 3 years?

(c) After how many years will the number of frogs reach 600?

$N(t) = 85e^{0.18t}$ N = number of frogs
t = time (yrs)

a) initial population → let $t=0$ $\therefore N = 85e^{0} = 85$

b) want N when $t=3$ $\therefore$ $N = 85e^{0.18(3)}$
$= 145.86$

$\therefore$ After 3yrs the population is 145

c) After how many years will population reach 600
ie: want t when $N=600$

$$\therefore 85e^{0.18t} = 600$$

$$e^{0.18t} = \frac{600}{85}$$

$$\therefore 0.18t = \ln\left(\frac{600}{85}\right)$$

$$t = \frac{\ln\left(\frac{600}{85}\right)}{0.18}$$

$$= 10.86$$

$\therefore$ Population reaches 600 in 10.86 years.

12. A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so that its temperature at time t is given by $T(t) = 18 + 62e^{-0.05t}$ where t is measured in minutes and T is measured in degrees Celsius(C).

(a) What is the initial temperature of the soup?

(b) What is the temperature after 10 minutes?

(c) After how long will the temperature be 37 degrees C?

(d) Make a graph of the temperature as a function of time.

$T(t) = 18 + 62e^{-0.05t}$

T = temp (°C)
t = time (mins)

a) initial temp → want T when $t = 0$

ie: $T = 18 + 62e^{0} = 18 + 62 = 80°C$

b) temp after 10 mins → want T when $t = 10$

$\therefore T(10) = 18 + 62e^{-0.05(10)}$

$= 55.6\ °C$

c) Now want time when Temp = 37

ie: want t when $T = 37$

$18 + 62e^{-0.05t} = 37$

$62e^{-0.05t} = 19$

$e^{-0.05t} = \frac{19}{62}$

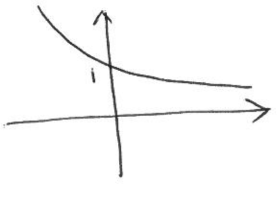
$\therefore -0.05t = \ln\left(\frac{19}{62}\right)$

$t = \frac{\ln\left(\frac{19}{62}\right)}{-0.05}$

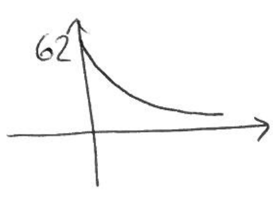
$= 23.65$

$\therefore$ It takes 23.65 mins for the temp to drop to 37°C

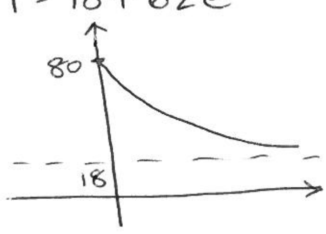
d) Now $T = e^{-0.05t}$



so $T = 62e^{-0.05t}$



so $T = 18 + 62e^{-0.05t}$



Half Life

- Remember for exponential decay

The halving time is constant

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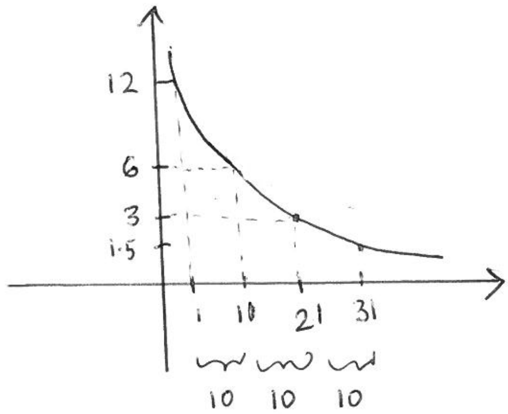
Time it takes to halve in value

↑

Half Life

eg



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Every 10 secs, the y-value halves

Here the half life = 10 secs.

- This concept is very useful for certain exponential decay models.

hosphorous has a half life of 14 days. Suppose a sample of this substance has a mass of 300 mg.

(a) Find the initial value and growth constant for this exponential decay function.

(b) Hence write down the exponential decay function which models the amount of Phosphorous that remains in the sample after t days.

We are told Phosphorus has a half life of 14 days

This is telling us we have an exp (decay) model.

So we expect model to be of the form

$P = P_0 e^{kt}$ (+ expect k to be neg)

P = amt of phos (mg)
t = time (days)

We are told the sample has mass of 300mg

ie: when $t=0$, $P=300$ ← initial value

$\therefore P_0 = 300$

so $P = 300e^{kt}$

Need to find k: We know half life is 14 days

ie: Mass drops to half its value in 14 days.

ie: when $P=150$, $t=14$

$\therefore$

$$150 = 300e^{14k}$$

$$\frac{150}{300} = e^{14k}$$

$$\frac{1}{2} = e^{14k}$$

$$\therefore 14k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln(1/2)}{14}$$

$$= -0.0495\ldots$$

$\therefore$ This model has equation $P(t) = 300e^{-0.0495t}$

Notice: I didn't need the initial amount to find the growth constant.

ie: $P = P_0 e^{kt}$

If the half life is 14 days

then when $t = 14$, $P = \frac{P_0}{2}$

So $\frac{P_0}{2} = P_0 e^{14k}$

$\therefore \frac{1}{2} = e^{14k}$

$\vdots$

etc

$k = -0.0495$

14. The half life of cesium-137 is 30 years. Suppose we have a 100 g sample.

(a) Find a function that models the mass remaining after t years.

(b) How much of the sample will remain after 4000 years?

(c) After how long will only 18 g of the sample remain?

a) Let cesium be modelled by the equation $C = C_0 e^{kt}$ $\left[\begin{array}{l} C = \text{cesium (g)} \\ t = \text{years} \end{array} \right.$

- initially sample $= 100g$ $\therefore C_0 = 100$
- half life $= 30$ years $\rightarrow$ when $t = 30$, $C = \frac{C_0}{2}$

$$\therefore \frac{C_0}{2} = C_0 e^{30k}$$

$$\therefore e^{30k} = \frac{1}{2}$$

$$30k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln(1/2)}{30} = -0.0231\ldots$$

So $C(t) = 100e^{-0.0231t}$

b) Want amount remaining after 4000 yrs.

ie: Want C when $t = 4000$

$$C = 100e^{-0.0231(4000)}$$

$$= 7.29 \times 10^{-39} \text{ mg}$$

c) Want time when 18g remains

ie: Want t when $C = 18$

$$\therefore 100e^{-0.0231t} = 18$$

$$e^{-0.0231t} = \frac{18}{100} = \frac{9}{50}$$

$$-0.0231t = \ln\left(\frac{9}{50}\right)$$

$$t = \frac{\ln\left(\frac{9}{50}\right)}{-0.0231} = 74.22$$

$\therefore$ It will take 74 yrs.

15. The half life of strontium-90 is 28 years. How long will it take a 50 mg sample to decay to a mass of 32 mg?

Strontium has half life of 28 yrs.

$\left[\begin{array}{l} P = \text{amt(mg)} \\ t = \text{years} \end{array} \right]$

$\therefore$ Let eqn be $P = P_0 e^{kt}$

Half life = 28 yrs $\rightarrow$ when $t = 28$, $P = \frac{P_0}{2}$

$\therefore$ Finding k :

$$\frac{P_0}{2} = P_0 e^{28k}$$

$$\frac{1}{2} = e^{28k}$$

$$= 28k.$$

$$k = \frac{\ln(1/2)}{28} = -0.0247\ldots$$

initially sample = 50 mg $\rightarrow$ $P_0 = 50$

so $P = 50 e^{-0.0247t}$

· want time to decay to 3 mg

$\rightarrow$ want t when $P = 32$

$$\therefore 50 e^{-0.0247t} = 32$$

$$e^{-0.0247t} = \frac{32}{50}$$

$$-0.0247t = \ln\left(\frac{32}{50}\right)$$

$$t = \frac{\ln\left(\frac{32}{50}\right)}{-0.0247}$$

$$= 18.03$$

$\therefore$ It takes $\approx$ 18 yrs.

16. A wooden artifact from an ancient tomb contains 65% of the carbon-14 that is present in living trees. The half-life of carbon-14 is 5730 years. How long ago was the artifact made?

First we'll find an eqn to model the decay : Let $C = C_0 e^{kt}$

half life = 5730 → when $t = 5730$ $C = \frac{C_0}{2}$

$$\therefore \frac{C_0}{2} = C_0 e^{5730k}$$

$$\frac{1}{2} = e^{5730k}$$

$$5730k = \ln(1/2)$$

$$k = \frac{\ln(1/2)}{5730}$$

$$\therefore C = C_0 e^{-0.00012t}$$

we want age of artifact

We know it has 65% of original amount.

ie: Want t when $C = 0.65 C_0$

$$\therefore 0.65 C_0 = C_0 e^{-0.00012t}$$

$$0.65 = e^{-0.00012t}$$

$$\ln(0.65) = -0.00012t$$

$$\therefore t = \frac{\ln(0.65)}{-0.00012}$$

$$= 3561.13$$

$\therefore$ The artifact was made $\approx$ 3561 yrs ago.